Birational Maps and Blowing Things Up

UoE Geometry Club

September 2006 Andrew Wilson

Contents

Introduction

Algebraic Geometry is the study of algebraic objects using geometrical tools. It can be said of algebraic geometry that the subject begins where equation solving ends. The main objects of study are algebraic varieties, that is, the sets of solutions to a collection of polynomials. One of the main problems, that serves as a yardstick of the progress made in this area, is that of classifying all varieties up to isomorphism. Here we discuss birational maps as our main topic, with the classification problem serving as our context. There will be lots on blowups, a bit about the difference between rational and regular maps, a little on resolving singularities (with blowups), plenty of examples and not too many technical details - a focus more on gaining an understanding of how all the machinery of algebraic geometry fits together and what use it can be put to ;-) We will start from the beginning, with very basic definitions – so no prior knowledge required. The ideas and much of the text here has been kidnapped from the books [Harris], [Harts] and [Shaf]. For more information they are your first recommended ports of call.

0.1. **The Classification Problem.** In its strongest form, the classification problem is; to classify, up to isomorphism, all algebraic varieties. As Hartshorne explains in [Harts] this problem can be broken down into three smaller problems;

(i) classify varieties up to birational equivalence (much coarser than isomorphism, as we'll see later)
(ii) identify a good subset of a birational equivalence class and classify up to isomorphism, for example, nonsingular projective varieties
(iii) study how far an arbitrary variety is from one of the nice varieties considered above. In particular;
   (a) how much do we need to add to a nonprojective variety to get a projective one and,
   (b) what is the structure of the singularities and how can they be resolved to give a nonsingular variety

Keeping this in mind, we shall remind ourselves of some of the basic ideas and definitions, in the next section. Then moving on to hopefully get a feel for the difference between birational geometry and other branches of geometry.

Firstly, as I always find it helpful, before diving in to definitions we give a historical sketch of the subject to show the development of the ideas used here.

0.2. **History.** Wikipedia [Wiki] says on the history of algebraic geometry;

> Algebraic geometry was largely developed by Muslim mathematicians, particularly the Persian mathematician/poet Omar Khayym (born 1048). He was well known for inventing the general method of solving cubic equations by intersecting a parabola with a circle. In addition he authored criticisms of Euclid's theories of parallels which made their way to England, where they contributed to the eventual development of non-Euclidean geometry. Omar Khayym also combined the

use of trigonometry and approximation theory to provide methods of solving algebraic equations by geometrical means.

Algebraic geometry was further developed by the Italian geometers in the early part of the 20th century. Enriques classified algebraic surfaces up to birational isomorphism. The style of the Italian school was very intuitive and does not meet the modern standards of rigor.

By the 1930s and 1940s, Oscar Zariski, Andr Weil and others realized that algebraic geometry needed to be rebuilt on foundations of commutative algebra and valuation theory. Commutative algebra (earlier known as elimination theory and then ideal theory, and refounded as the study of commutative rings and their modules) had been and was being developed by David Hilbert, Max Noether, Emanuel Lasker, Emmy Noether, Wolfgang Krull, and others. For a while there was no standard foundation for algebraic geometry.

In the 1950s and 1960s Jean-Pierre Serre and Alexander Grothendieck recast the foundations making use of sheaf theory. Later, from about 1960, the idea of schemes was worked out, in conjunction with a very refined apparatus of homological techniques. After a decade of rapid development the field stabilised in the 1970s, and new applications were made, both to number theory and to more classical geometric questions on algebraic varieties, singularities and moduli.

An important class of varieties, not easily understood directly from their defining equations, are the abelian varieties, which are the projective varieties whose points form an abelian group. The prototypical examples are the elliptic curves, which have a rich theory. They were instrumental in the proof of Fermat's last theorem and are also used in elliptic curve cryptography.

While much of algebraic geometry is concerned with abstract and general statements about varieties, methods for effective computation with concretely-given polynomials have also been developed. The most important is the technique of Gröbner bases which is employed in all computer algebra systems.

1. Preliminaries

Lettuce start here by running through some of the basic definitions and results to get us quickly to the next sections where the main meal for the day is. The following is taken, on the most part from [Shaf].

1.1. **Spaces.**

Definition 1. We write $\mathbb{A}^n$ for the n-dimensional *affine space* over the field k. Its points are of the form $(\alpha_1, \ldots, \alpha_n)$. The main distinction between $\mathbb{A}^n$ and k^n as a vector space is that the origin plays no special role here.

Definition 2. A *closed subset* of $\mathbb{A}^n$ is a subset $X \subseteq \mathbb{A}^n$ consisting of all common zeros of a finite number of polynomials with coefficients in k. A closed subset defined by just one polynomial is called a *hypersurface*.

Definition 3. We write $\mathbb{P}^n$ for the $(n+1)$-dimensional *projective space* over the field k. It consists of all the one dimensional subspaces of the vector space k^{n+1}. Its points we write as homogeneous vectors $(\alpha_0 : \alpha_1 : \cdots \alpha_n)$, denoting the line spanned by $(\alpha_0, \alpha_1, \ldots \alpha_n) \in k^{n+1}$.

Definition 4. A *closed subset* of $\mathbb{P}^n$ is a subset $X \subseteq \mathbb{P}^n$ consisting of all common zeros of a finite number of *homogeneous* polynomials with coefficients in k. Where a *homogeneous polynomial* or *form* is a polynomial whose terms are monomials all having the same total degree (e.g. $x^5 + 3xy^4 - 2x^3y^2$ is a homogeneous polynomial of degree 5 in two variables). A closed subset defined by just one homogeneous polynomial is called a *hypersurface*.

There is the notion of *quasi-projective* variety which unites the concepts of affine and projective closed subsets. We shall skirt around the details here and talk of *varieties* (affine or projective) as these closed subsets in the above definitions.

Before we move on to define morphisms between these spaces, let's look at at one further concept.

Definition 5. A closed set X is *reducible* if there exists proper closed subsets $X_1, X_2 \subsetneq X$ such that $X = X_1 \bigcup X_2$. Otherwise X is *irreducible*.

Finally, we note a few of the properties of irreducible sets:

- Any closed set X is the union of irreducible closed sets
- If $X = \bigcup X_i$ is an expression of X as a finite union of irreducible closed sets, and if $X_i \subset X_j$ for $i \neq j$ then we may delete X_i from the expression. Repeating this process, we arrive at an expression $X = \bigcup X_i$ in which $X_i \not\subset X_j$ for all $i \neq j$. We say that such a representation of X is *irredundant* and that the X_i are the *irreducible components* of X
- An irredundant representation of a closed set X is unique
- A product of irreducible closed sets is irruducible
- If Y is an irreducible subset of X, then its closure $\bar{Y}$ in X is also irreducible

1.2. **Morphisms.** There are subtle differences between the definitions of functions and maps for affine and projective varieties. We'll first define them for the affine case.

1.2.1. *Morphisms in Affine Space.*

Definition 6. For $X \in \mathbb{A}^n$, a *regular function* $f : X \to k$ is an everywhere-defined, polynomial function on an affine or projective variety X taking values in the field k.

The set of all regular functions on a given variety X forms a ring, with addition and multiplication defined as for polynomials. This is the *coordinate ring*, denoted $k[X]$. With each polynomial $F \in k[T_1, \dots, T_n]$ we may associated a function $\varphi \in k[X]$, by viewing F as a function on the points of X. Thus, we have a homomorphism between the rings $k[T_1, \dots, T_n]$ and $k[X]$. The kernel of this homomorphism consist of all the polynomials that take the value zero at all points of X. We call this ideal *the ideal of the closed set X* and denote it $\mathfrak{U}_X$. We see from this that

$$k[X] = k[T_1, \dots, T_n]/\mathfrak{U}_X$$

If we take an arbitrary ring R, can it be thought of as a coordinate ring for some space X, that is does $R = k[X]$? The answer; yes, if the ring has no nilpotents and is finitely generated as an algebra over k.

Definition 7. For $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, a map $\varphi : X \to Y$ is *regular* if there exist m regular functions $f_1, \dots, f_m$ on X such that $\varphi(x) = (f_1(x), \dots, f_m(x))$ for all $x \in X$.

Aside. Some remarks on arbitrary maps between sets. If $\varphi : X \to Y$ is a map between sets X and Y, then for every function u on Y (taking values in some set Z) we associate a function v on X by taking $v(x) = u(\varphi(x))$. We set $v = \varphi^*(u)$ and call it the *pullback* of u. Thus, φ^* maps functions on Y to functions on X.

For $\varphi : X \to Y$ a regular map between closed sets X and Y, the pullback map gives a homomorphism between the coordinate rings, that is, $\varphi^* : k[Y] \to k[X]$. Notice that the kernal of φ^* is zero if and only if $\varphi(X)$ is dense in Y. In this case φ^* is an isomorphic inclusion $\varphi^* : k[Y] \hookrightarrow k[X]$.

The above gives us the notion of when two affine varieties are the same, namely, if there exist maps $\varphi : X \to Y$ and $\chi : Y \to X$ inverse to one another in both directions, or equivalently $k[X] \cong k[Y]$. We say is this case that X and Y are *isomorphic* or *biregular*. Thus the coordinate ring of a variety is an invariant. Relationships between varieties are often reflected in the algebra, as another example; if X and Y are closed subsets of $\mathbb{A}^n$ then,

$$Y \subset X \iff \mathfrak{U}_X \subset \mathfrak{U}_Y$$

Definition 8. For an irreducible set X, the field of fractions of the coordinate ring $k[X]$ is the *function field*, or the *field of rational functions* of X; it is denoted $k(X)$.

Thus, a rational function $f \in k(X)$ can be written g/h with $g, h \in k[X]$ and $h \notin \mathfrak{U}_X$. $g/h = g'/h'$ if $gh' - hg' \in \mathfrak{U}_X$. Thus, we may also construct $k(X)$ as follows. Consider the subring $\mathcal{O} \subseteq k(T_1, \dots, T_n)$ of rational functions $f = g/h$ with $g, h \in k[T_1, \dots, T_n]$ and $h \notin \mathfrak{U}_X$. The functions in $\mathcal{O}$ with $g \in \mathfrak{U}_X$ form an ideal, M_X and we have

$$k(X) = \mathcal{O}/M_X$$

Notice that these are not really functions at all! They do not necessarily have well-defined values at all points of X.

Some points to note about rational functions:

- A rational function $f \in k(X)$ is regular at $x \in X$ if it can be written in the form $f = g/h$ with $g, h \in k[X]$ and $h(x) \neq 0$. In this case we say that the element $g(x)/h(x) \in k$ is the value of f at x, and denote it by $f(x)$
- A rational function that is regular at all points of a closed subset X is a regular function on X

Definition 9. For $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, a *rational map* $\varphi : X \dashrightarrow Y$ is an m-tuple of rational functions $f_1, \ldots, f_m \in k(X)$, such that for all points that the f_i are regular $\varphi(x) = (f_1(x), \ldots, f_m(x)) \in Y$. We say that φ is regular at x and that $\varphi(x) \in Y$ is the *image* of φ.

1.2.2. *Morphisms in Projective Space.* What about functions and maps on projective space? The problem is that a rational function

$$f(x_0, \ldots, x_n) = \frac{g(x_0, \ldots, x_n)}{h(x_0, \ldots, x_n)}$$

on homogeneous coordinates $x_0, \ldots, x_n$ is not a function of $(\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}^n$, even when $h(\alpha_0, \ldots, \alpha_n) \neq 0$, as in general the values of f will change (as $(\alpha_0 : \cdots : \alpha_n) = (\lambda\alpha_0 : \cdots : \lambda\alpha_n)$). The solution to this is to take f to be a homogeneous function of degree 0, that is take g and h to be homogeneous polynomials of the same degree. We state this in the following.

Definition 10. For a projective variety $X \subseteq \mathbb{P}^n$ with $x \in X$, a homogeneous function of degree 0, $f = g/h$ with $h(x) \neq 0$ defines a function in a neighbourhood of x taking values in the field k. We say that f is *regular* at x. If f is regular at every point of X, then we say that f is a *regular function*. All regular functions on a variety X form a ring denoted $k[X]$.

The ring $k[X]$ is no longer a useful invariant when we speak of projective varieties. For any irreducible closed projective set X, $k[X]$ consists only of constants. To see this in the case where $X = \mathbb{P}^n$; if $f = g/h$ is a regular function on $\mathbb{P}^n$, with g and h forms of the same degree, where we may assume that g and h have no common factors. Then f is not regular where $h(x) = 0$ and so g and h are constants.

Definition 11. For an irreducible projective variety $X \subseteq \mathbb{P}^n$, a *regular map* $\varphi : X \to \mathbb{P}^m$ is given by an $(m+1)$-tuple of forms $(f_0 : \cdots : f_m)$ of the same degree in the homogeneous coordinates of $x \in \mathbb{P}^n$. We require that for every $x \in X$ at least one of the $f_i(x) \neq 0$. Two maps $\varphi(x) = (f_0 : \cdots : f_m)$ and $\chi(x) = (g_0 : \cdots : g_m)$ are considered to be equal when $f_i g_j = f_j g_i$ on X $(0 \leq i, j \leq m)$. We say that φ maps X to Y, for some subset $Y \subseteq \mathbb{P}^m$, if $\varphi(X) \subseteq Y$.

Definition 12. For an irreducible projective variety $X \subseteq \mathbb{P}^n$, write $\mathcal{O}_X$ for the ring of rational functions $f = g/h$ with g and h forms of the same degree and $h \notin \mathfrak{U}_X$. Writing M_X for the set of functions $f \in \mathcal{O}_X$ with $h \in \mathfrak{U}_X$, we have the quotient ring $\mathcal{O}_X/\mathfrak{U}_X$ which is a field, as M_X is a maximal ideal. We call this field the *function field* or the *field of rational functions* and denote it by $k(X)$.

Definition 13. As in the definition of a regular map, for an irreducible projective variety $X \subseteq \mathbb{P}^n$, a *rational map* $\varphi : X \dashrightarrow \mathbb{P}^m$ is given by an $(m+1)$-tuple of forms $(f_0 : \cdots : f_m)$ of the same degree in the homogeneous coordinates of $x \in \mathbb{P}^n$. We require that at least one of the forms does not vanish on X. Two maps $\varphi(x) = (f_0 : \cdots : f_m)$ and $\chi(x) = (g_0 : \cdots : g_m)$ are considered to be equal when $f_i g_j = f_j g_i$ on X $(0 \leq i, j \leq m)$. The set of points on which a rational map is regular is open. Thus, we may say that a rational map is a regular map of some open set $U \subset X$. We say that φ maps X to Y, for some subset $Y \subseteq \mathbb{P}^m$, if $\varphi(U) \subseteq Y$.

Example 14. The Segre embedding.
For $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, we want to consider the product space $X \times Y$ as a projective variety. To this end we construct an embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^N$ for some N.

Let $\mathbb{P}^N$ have homogeneous coordinates w_{ij} with $i = 0, \ldots, n$ and $j = 0, \ldots, m$, so that $N = (n+1)(m+1) - 1$. We define $\varphi : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ by sending $(x_0 : \cdots : x_n; y_0 : \cdots : y_m)$ to the point $(w_{ij} = x_i y_j)$. Restricting this embedding to $X \times Y$ we have the product space realised as a projective variety.

1.3. **Singular Points.** Before moving on to the next section let's examine one final concept that we'll need later on: singular points. Instead of defining a singular point in purely algebraic terms, via dimensions of tangent spaces, we give a pair of definitions for determining if a point on a hypersurface is singular; one each for the affine and projective cases.

Definition 15. For an affine hypersurface in $\mathbb{A}^n$ defined by the single equation $f(x_1, \ldots, x_n) = 0$, the point $\alpha \in \mathbb{A}^n$ is *singular* if both $f(\alpha)$ and the partial derivatives evaluated at α have a common solution, that is,

$$f(\alpha) = \frac{\partial f}{\partial x_i}_{|(\alpha)} = 0$$

The definition for the projective case is ostensibly the same, the main difference being the existance of a theorem of Euler's on homogenous functions ($x_i \frac{\partial f}{\partial x_i} = n f(x)$). Therefore, we no longer need the condition that $f(\alpha) = 0$.

Definition 16. For a projective hypersurface in $\mathbb{P}^n$ defined by the single homogeneous equation $f(x_0, x_1, \ldots, x_n) = 0$, the point $\alpha \in \mathbb{P}^n$ is *singular* if the partial derivatives evaluated at α have a common solution, that is,

$$\frac{\partial f}{\partial x_i}_{|(\alpha)} = 0$$

2. Birational Maps

This section is, for me, where we start to see the theory pooling together. Let's dive straight in. Throughout X,Y and Z are varieties.

2.1. **Composing Rational Maps.** If we are to speak of birational maps, then we had better understand how to compose two maps first. The problem raises its head, since as we have seen a rational map is no map at all. It may be the case that for $\chi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ the image of X under χ consists only of undefined points of the map ψ and hence the composition is undefined, even as a rational map. For example, consider the maps $\chi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ and $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by $(x : y : z : t) \mapsto (x : y : 0 : 0)$ and $(x : y : z : t) \mapsto (0 : 0 : z : t)$ respectively. Then what of the composition $\psi \circ \chi$? Argg...it's undefined everywhere – that won't do!

To resolve this issue let's look at a new definition for a rational map.

Definition 17. A *rational map* is an equivalence class of pairs $\langle U, \varphi_U \rangle$ where U is an open subset of X and φ_U is a regular map from U to Y. Two pairs $\langle U, \varphi_U \rangle$ and $\langle V, \varphi_V \rangle$ are considered to be equal if φ_U and φ_V agree on $U \cap V$.

Now, for two rational maps $\chi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ we take representatives of both maps $\langle U, \chi_U \rangle$ and $\langle V, \psi_V \rangle$ and when $\chi_U^{-1}(V)$ is nonempty we define the composition $\psi_V \circ \chi_U$ of χ_U and ψ_V to be the equivalence class $\langle \chi_U^{-1}(V), \psi_V \circ \chi_U \rangle$.

In practice to work with rational maps we think of them as in Definition **??**, that is, as a collection of rational functions. Definition 17 however gives us a second way of viewing them: a rational map $\varphi : X \dashrightarrow Y$ is a regular map on an open subset of X.

Clearly we would like to talk about surjective maps. On recalling the definition of a surjective map, we see that a rational map cannot, in general, be surjective. Indeed, for a rational map $\varphi : X \dashrightarrow Y$ we require for surjectivity that for each $y \in Y$ there exists $x \in X$ such that $y = \varphi(x)$. Thus, we make the following definition which we may think of as surjectivity in the category of projective spaces and rational maps.

Definition 18. A rational map $\varphi : X \dashrightarrow Y$ is *dominant* if for some representative $\langle U, \varphi_U \rangle$, and hence all, the image of φ_U is dense in Y.

It should be clear from the definition that any two dominant rational maps are composible. To sum up, to make ensure two maps are compatible, we may either confirm that the codimension of the images of both maps are not greater than two or alternatively work with dominant rational maps.

Let's look again at the algebra side of rational maps. We know that if there is a rational map between to varieties X and Y that the pullback defines a map between the function fields of X and

Y. More precisely, for a rational map $\varphi : X \dashrightarrow Y$ we have the pullback map $\varphi^* : k(Y) \to k(X)$.

Suppose that φ is dominant, then for any $f \in k(Y)$, $\varphi^*(f)$ is a well defined rational function on X. Thus, $\varphi^* : k(Y) \hookrightarrow k(X)$ is an isomorphic inclusion.

This leads us to the following theorem;

Theorem 19. *For any two varieties X and Y, we have a bijection (given by the pullback map) between*

- *the set of dominant rational maps from X to Y, and*
- *the set of k-algebra homomorphisms from $k(Y)$ to $k(X)$*

That is the category of projective varieties and dominant rational maps is equivalent to the category of finitely generated field extensions of k with the arrows reversed. Thus, in the following we shall think of all our rational maps to be dominant.

2.2. **Birational Maps.** Without further ado;

Definition 20. A *birational map* $\varphi : X \dashrightarrow Y$ is a rational map that admits an inverse. That is, there exists a rational map $\psi : Y \dashrightarrow X$ such that ψ and φ are composible and $\psi \circ \varphi = id_X$, $\varphi \circ \psi = id_Y$. We say in this case that X and Y are *birationally equivalent* or that X and Y are *birational*.

The notion of birationality is an equivalence relation. This gives us our first step towards the classification problem. From what we have discussed above, we may identify when two varieties are birational in several ways, which we state in the following theorem.

Theorem 21. *The following statements are equivalent*

- *X and Y are birational*
- $K(X) \cong K(Y)$
- $\exists$ *open subsets U,V of X and Y, respectively such that $U \cong V$*

To bring some of the above into greater focus, we shall look next at a specific example.

Example 22. The Quadratic Surface.
One of the simplest examples of a birational equivalence is that between a quadratic surface in $\mathbb{P}^3$ and $\mathbb{P}^2$. More specifically, let's take the quadric surface $Q \in \mathbb{P}^3$ defined by $xt - yz = 0$ and work through the details of Theorem 21.

Firstly, consider the projection of Q from the point $p = (0:0:0:1)$ to $\mathbb{P}^2$ this is the map $\pi_p : Q - \{p\} \to \mathbb{P}^2$ that maps the point $(x:y:z:t)$ to $(x:y:z)$. This defines a rational map $\pi : Q \dashrightarrow \mathbb{P}^2$. Clearly $im(Q) = \mathbb{P}^2$. Define $\pi^{-1} : \mathbb{P}^2 \dashrightarrow Q$ by $(x:y:z) \mapsto (x^2 : xy : xz : yz)$. Then

π^{-1} is dominant and we have $\pi \circ \pi^{-1} = id_{\mathbb{P}^2}$ and $\pi^{-1} \circ \pi = id_(Q)$.

Secondly, observe that the Serge embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is Q. Thus, $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and since $k(\mathbb{P}^1 \times \mathbb{P}^1) = k(x,y)$ we have by Theorem 21 that Q is birational to $\mathbb{P}^2$.

Finally, we may observe that the open subset $U \in Q$ defined by $U = \{(x:y:z:t) : x \neq 0\}$ is isomorphic to $\mathbb{A}^2$ by the maps $\xi : U \to \mathbb{A}^2, (x:y:z:t) \mapsto (y:z)$ and $\xi^{-1} : \mathbb{A}^2 \to U, (y:z) \mapsto (1:y:z:yz)$. Since $\mathbb{A}^2$ is an open subset of $\mathbb{P}^2$, we have a third time that Q is birationally equivalent to $\mathbb{P}^2$.

Definition 23. We say that a given variety is *rational* if it is birational to $\mathbb{P}^n$ for some n.

Thus, from the previous example, Q is a rational surface or put another way both Q and $\mathbb{P}^2$ lie in the same birational equivalence class.

3. The Blowup Map

The blowup map (think: balloon expanding – rather than dynamite exploding) is the typical example of a birational map that is not an isomorphism (in general), in fact it is a regular birational map. It also serves as the main tool in resolving singularities, which we shall see later in Section 4. We shall work up to the full definition of the blowup piece-by-piece. While the blowup is very easy to define, it is not easy to describe what it looks like in general.

3.1. **Blowup at a Point.** Firstly, let's look at the blowup of $\mathbb{A}^n$ at the point $(0,\ldots,0)$.

Consider the quasi-projective variety $\mathbb{A}^n \times \mathbb{P}^{n-1}$. Let $(a_1,\ldots,a_n)$ be the affine coordinates of $\mathbb{A}^n$ and $(y_1,\ldots,y_n)$ be the homogeneous coordinates of $\mathbb{P}^{n-1}$. Then any closed set of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ is defined by polynomials in the a_i and y_j, homogeneous in the y_j.

Definition 24. The *blowup* of $\mathbb{A}^n$ at the point $P = (0,\ldots,0)$ is the subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $\{x_i y_j = x_j y_i : i,j = 1,\ldots,n\}$.

We also have a natural map, $\sigma : X \to \mathbb{A}^n$ defined by restricting the projection of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ onto the first factor. We have then, a picture that looks like;

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\sigma"'] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d] \\ & \mathbb{A}^n \end{tikzcd}$$

Before we look at the properties of this blowup and the blowup at a subvariety of $\mathbb{A}^n$, let's have a look at a concrete example.

Example 25. Consider the cusp C in $\mathbb{A}^2$, given by the equation $y^2 = x^3$. We shall blowup C at the point $p = (0,0)$.

Let $\mathbb{A}^2$ have the affine coordinates (x,y) and $\mathbb{P}^1$ the homogeneous coordinates (s,t). Then the blowup X of $\mathbb{A}^2$ is given by the equation $xt = ys$. X here looks like $\mathbb{A}^2$, except the point p has been replaced by $\mathbb{P}^1$. To obtain the total inverse image of C in X we consider the equations $y^2 = x^3$ and $xt = ys$ in $\mathbb{A}^2 \times \mathbb{P}^1$. Since the coordinates of $\mathbb{P}^1$ are homogeneous, either s or t is nonzero. Suppose $s \neq 0$ then we may assume that $s = 1$ and we then have the equations $y^2 = x^3$ and $y = xt$. Substituting, we have; $x^2(x-t) = 0$. Hence the total inverse image of C is composed of two irreducible components; one defined by $x = 0$, $y = 0$, t arbitrary, which we will call E and the other defined by $x = t^2$, $y = xt$, this is the blow up of C, which we denote here by $\widetilde{C}$. Pictorially this looks something like Figure 25.

Some properties to observe

- away from the point P, we have an isomorphism between C and $\widetilde{C}$, given by
$$(x,y) \mapsto (x,y;\frac{y}{x}) \qquad \text{and} \qquad (x,y;t) \mapsto (x,y)$$
- $\sigma^{-1}(P) \cong \mathbb{P}^1$
- the points of $\sigma^{-1}(P)$ are in one-to-one correspondence with the set of lines through P.

  To see this; a line L in $\mathbb{A}^2$ passing through P and an arbitrary point (a_1,a_2), with a_1,a_2 not both zero, can be described by the parametric equations $(a_1\lambda, a_2\lambda)$, with $\lambda \in \mathbb{A}^1$. Consider now, the line $L' = \sigma^{-1}(L-P)$, assuming $a_2 \neq 0$, L' is given by
$$(a_1\lambda, a_2\lambda; \frac{a_1\lambda t}{a_2\lambda} : t) = (a_1\lambda, a_2\lambda; a_1 : a_2)$$
  , where $\lambda \neq 0$ since we're not considering the point P. Notice that $(a_1\lambda, a_2\lambda; a_1 : a_2)$ also makes sense when $\lambda = 0$, this will give us the closure $\bar{L}'$ of L' in X. $\bar{L}'$ and $\sigma^{-1}(P)$ meet at the point $(0,0;a_1 : a_2)$.
- X is irreducible.

  X is the union of $X - \sigma^{-1}(P)$ and $\sigma^{-1}(P)$, $X - \sigma^{-1}(P)$ is isomorphic to $\mathbb{A}^2 - P$ and so irreducible. Also, we have seen above that every point of $\sigma^{-1}(P)$ belongs to the closure of a line in $X - \sigma^{-1}(P)$. Thus $X - \sigma^{-1}(P)$ is dense in X and, as we noted in Section 1.1, thus we have that X is irreducible.

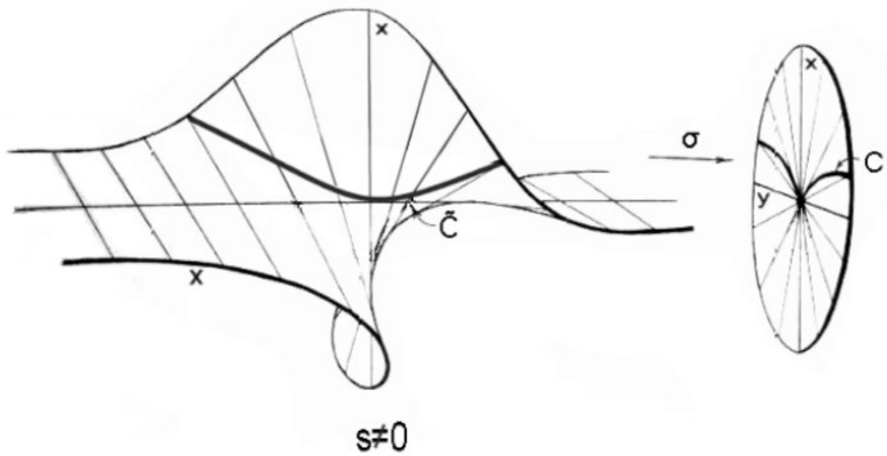

Figure 1. Blowup of the cusp $y^2 = x^3$ at (0,0) on the chart $s \neq 0$ (image thanks to [Harts])

Next we'll give the definition of the blowup of a variety at a point, as in Example 25 above.

Definition 26. If Y is a closed subvariety of $\mathbb{A}^n$ passing through $P = (0,\dots,0)$, we define the *blowup of Y at the point P* to be $\widetilde{Y} = \overline{\sigma^{-1}(Y-P)}$, where $\sigma : X \to \mathbb{A}^n$ is the blowup of $\mathbb{A}^n$ at the point P descibed in Definition 24. We denote also by $\sigma : \widetilde{Y} \to Y$ the restriction of $\sigma : X \to \mathbb{A}^n$ to $\widetilde{Y}$. To blowup Y at any other point Q, make a linear change of coordinates sending P to Q.

3.2. **More Blowups.** So far the blowups we have looked at have all been blowups at a point. We may also blowup at a collection of points, or even along a subvariety. We may also blowup many points, or even blowup along a subvariety. In the case of blowing up many points, the blowup will be the same whether we blowup at all points simultaneously or blowup at one point, then blowup the next point on the blowup, etc.. As the purpose of this exposition is to give a feel for birational geometry, rather than all its technical details, we have chosen not to include this concept.

The following theorem will give us yet another point of view of rational maps; a rational map $\varphi : X \to \mathbb{P}^n$ is a regular map on a blowup of X. Hence to understand rational maps we need only understand regular maps and blowups.

Theorem 27. *Let $\varphi : X \dashrightarrow \mathbb{P}^n$ be any rational map. Then φ can be resolved as a sequence of blowups, that is, there is a sequence of varieties $X = X_1, X_2, \dots, X_k$, subvarieties $Y_i \subseteq X_i$ and maps $\pi_i : X_{i+1} \to X_i$ such that*

(i) *$\pi_i : X_{i+1} \to X_i$ is the blowup of X_i along Y_i and*

(ii) *the map φ factors into a composition $\widetilde{\varphi} \circ \pi_k^{-1} \circ \cdots \circ \pi_1^{-1}$ with $\widetilde{\varphi} : X_{k+1} \to \mathbb{P}^n$ regular.*

$$\begin{tikzcd}
X_{k+1} \arrow[d, "\pi_k"'] \arrow[ddddr, "\widetilde{\varphi}"] & \\
X_k \arrow[d] & \\
\vdots \arrow[d] & \\
X_2 \arrow[d, "\pi_1"'] & \\
X \arrow[r, dashed, "\varphi"'] & \mathbb{P}^n
\end{tikzcd}$$

Put in another way, after we blowup X up finite number of times, we arrive at a variety $\widetilde{X}$ with X and $\widetilde{X}$ birational such that the induced rational map $\widetilde{\varphi} : X_{k+1} \to \mathbb{P}^n$ is in fact regular.

Example 28. As an example of this resolution of a rational map into a sequence of blowups, let's look at resolving the map $\pi : Q \dashrightarrow \mathbb{P}^2$ from Example 22.

Firstly, we define the surface $\Gamma \subseteq Q \times \mathbb{P}^2$ to be the surface consisting of all the points $(x : y : z : t; x : y : z)$ with $(x : y : z : t)$ in Q. This is the *graph* of the map $\pi : Q \dashrightarrow \mathbb{P}^2$.

We claim that Γ is the blowup of Q at the point $(0 : 0 : 0 : 1)$.

Before proving this claim, note that the map $\psi : \Gamma \to \mathbb{P}^2$, given by $(x : y : z : t; x : y : z) \mapsto (x : y : z)$ is regular.

Proof of claim:

Let $\widetilde{Q}$ be the blowup of Q at the point $(0 : 0 : 0 : 1)$. Then we have the following picture

$$\begin{tikzcd}
\widetilde{Q} \arrow[r, hook] \arrow[dr, "\sigma"'] & Q \times \mathbb{P}^2 \arrow[d] \\
& Q
\end{tikzcd}$$

where σ is the blowup map. $\widetilde{Q}$ is defined as the set

$$\{(x : y : z : t; \alpha : \beta : \gamma) : xt - yz, x\beta = y\alpha, x\gamma = z\alpha, y\gamma = z\beta\}$$

- $\Gamma \subseteq \widetilde{Q}$: Let $(x : y : z : t; x : y : z) \in \Gamma$. Then $xt = yz$ and $\alpha = x, \beta = y, \gamma = z$, so that the equations $x\beta = y\alpha, x\gamma = z\alpha, y\gamma = z\beta$ are all satisfied.

- $\widetilde{Q} \subseteq \Gamma$: Let $q \in \widetilde{Q}$, then $q = (x:y:z:t;\alpha:\beta:\gamma)$ with $xt = yz, x\beta = y\alpha, x\gamma = z\alpha, y\gamma = z\beta$. We prove that on each of the three charts defined by $x \neq 0$, $y \neq 0$, $z \neq 0$ we have $x = \alpha, y = \beta, z = \gamma$ (we need not consider the chart on which $t \neq 0$ as $(0:0:0:1;0:0:0)$ is not a point of either Γ or $\widetilde{Q}$.

  Suppose $x \neq 0$, then we may assume that $x = 1$ and q is then the point $(1:y:z:t;1:\alpha:\beta:\gamma)$. The equations $t = yz, \beta = y\alpha, \gamma = z\alpha, y\gamma = z\beta$ hold here, thus we may write $q = (1:y:z:yz;\alpha:y\alpha:z\alpha) = (1:y:z:yz;1:y:z)$. Hence we have, on the chart $x \neq 0$, $x = \alpha, y = \beta, z = \gamma$. Similarly on the remaining two charts.

$\square$

Relating back now to Theorem 27, we have resolved the rational map $\pi : Q \dashrightarrow \mathbb{P}^2$ into the composition of a blowup from Q to Γ and a regular map from Γ to $\mathbb{P}^2$ looking something like

$$\begin{tikzcd} \Gamma \arrow[d, "\sigma"'] \arrow[dr, "\psi"] & \\ Q \arrow[r, dashed, "\pi"'] & \mathbb{P}^2 \end{tikzcd}$$

So far, we have seen an overview of what makes rational and birational maps tick. This is a good step towards understanding the first of Hartshornes goals for the classification problem, that is; classifying all varieties up to birational equivalence.

We go on now to look at his second goal, namely, identifying a 'good' subset of a birational equivalence and classify up to isomorphism.

4. Resolution Of Singularities

One such 'good' subset of a given birational equivalence class is that of the nonsingular or smooth varieties. For example, if we are working over a field of characteristic 0, we may throw all the power of complex manifold theory at our problem. However, given an arbitrary variety X, does there exist a nonsingular variety Y birational to it? This is known to be true in characteristic 0 for all varieties, but is only known to be true in characteristic p for curves and surfaces. Still, it seems a reasonable place to start.

We may state this in the following theorem.

Theorem 29. *Let X be any variety. Then there exists a nonsingular variety Y and a regular birational map $\pi : X \to Y$.*

As we know from the previous section, the map π can be resolved as a sequence of blowups. We see then that blowups are our main tool for the resolution of singularities. We have already looked at an example of this, in Example 25 we blew-up the cusp, C in $\mathbb{A}^2$, given by the equation $y^2 = x^3$ at the point $(0,0)$. The cusp C is singular at the point $(0,0)$ and nonsingular everywhere else. The blowup of C, $\widetilde{C}$ is isomorphic to C outside the point $(0,0)$, and the inverse image of $(0,0)$ on $\widetilde{C}$ is isomorphic to $\mathbb{P}^1$. Therefore, $\widetilde{C}$ is nonsingular.

5. Further Directions

Before completing this exposition, let's talk through a couple of directions we could take this theory further.

5.1. **Minimal Models.** We look again at the second of Hartshorne's problems towards the classification problem; that of finding a 'good' subset of a birational equivalence class. We say that a representative of a given birational equivalence class is a *model*. As we saw in the previous section, for most varieties we can find a nonsingular projective model in each class. What about the uniqueness of such a model? To discuss this, we should make a definition to help us explore the relations between models.

Definition 30. For two models X and X', we say that X' *dominates* X if there exists a regular birational map $\varphi : X' \to X$. A *minimal model* is a model such that any model which it dominates is in fact isomorphic to it.

There is a result which tells us that every variety dominates at least one minimal model. It can be easily shown that both the quadratic surface and $\mathbb{P}^2$ are minimal models and as we have seen they both lie in the same birational equivalence class. Thus we cannot hope for the uniqueness of minimal models in all dimensions. However, in the world of curves, we have a rather nice theorem; for nonsingular projective curves birationality is equivalent to isomorphism. Thus, nonsingular projective models are unique and the classification problem reduces to studying isomorphisms between curves. As it turns out, we may also find a unique minimal model in the world of surfaces – if we exclude certain types of surfaces; namely, rational surfaces and ruled surfaces (a ruled surface is a surface birational to $C \times \mathbb{P}^1$, where C is an algebraic curve).

5.2. **Unirationality.** Just to shake things up a bit let's end with a new concept that may prove to be a better generalisation to higher dimensions than that of birationality.

Definition 31. A variety X is *unirational* if there exists a dominant map $\varphi : \mathbb{P}^n \dashrightarrow X$, that is X is covered by a rational variety. Or, equivalently, if $k(X)$ embeds in a purely transcendental extension $k(x_1, \ldots, x_n)$ of k.

There is a classical theorem of the Italian school of algebraic geometry of Lüroth, that an algebraic curve is rational if and only if it is unirational. Or in algebraic terms a subfield of the field $k(x)$ that contains k and is not equal to k is in fact isomorphic to $k(x)$. Castelnuovo and Enriques proved the same for sufaces. Half a century later Clemens and Griffiths proved that most cubic three-folds are unirational but not rational. Thus, it looks like this notion may porve to be the better generalisation.

References

[Harts] Robin Hartshorne, *Algebraic Geometry*, Spinger, 1977
[Shaf] Igor R. Shafarevich, *Basic Algebraic Geometry 1*, Springer-Verlag, 1988
[Harris] Joe Harris, *Algebraic Geometry, A First Course*, Spinger-Verlag, 1992
[Wiki] *Wikipedia, the free encylopedia*, en.wikipedia.org

ANDREW WILSON

www.explicitlyambiguous.co.uk
in.one.ear.and.out.the.other@gmail.com

UNIVERSITY OF EDINBURGH SCHOOL OF MATHEMATICS, ROOM 5316, JCMB, KING'S BUILDINGS,
EDINBURGH, EH9 3JZ