1S Calculus

Chapter 2

University of Glasgow

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Ordinary differential equations

Definition

An *ordinary differential equation* (ODE) is an equation involving:

- an independent variable x,
- a dependent variable $y(x)$, and
- at least one of the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$.

If $\frac{d^ny}{dx^n}$ is the highest order derivative appearing in the equation, then the equation is an n-*th order* ordinary differential equation.

Example

For example, $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = x^4$ is a 2nd order ODE.

Ordinary differential equations

Definition

Let $f, g \in \{y, y', y'', \ldots\}$. An ODE is *linear* if it satisfies the following:

i) f occurs only in the 1^{st}–degree (i.e. no y^r or $(y^{(n)})^r$).

ii) No products $f \cdot g$ appear in the equation.

iii) No transcendental function of f occurs.

An ODE not satisfying the above conditions is called *non–linear*.

Example (Orders and linearity)

$y'' + y = x^2$	linear	second-order
$xy''' + (y')^2 - y \sin x = \log x$	nonlinear	third-order
$(y')^5 + y^2 = 0$	nonlinear	fifth-order
$y'' + \sin(y) = 0$	nonlinear	second-order
$p(x)y' + q(x)y = 0$	linear	first-order
$y' + z = x$ $z'' - y = \sin x$	a system of coupled ODEs	second-order

Example (Particular and general solutions)

i) For the linear ODE $\dfrac{d^2y}{dx^2} + y = 0$, $y = \sin x$ is **a** solution. However, it is not the only solution — $y = \cos x$ is also a solution. These are called **particular solutions (PS)**. In fact, for constants a, b any linear combination, $a \sin x + b \cos x$, of these particular solutions is a solution, called the **general solution (GS)**.

ii) Solve $y'' = 20x^3$.

Remarks

i) *If we give c_1 and c_2 particular values we get a* **PS**. *Every PS is obtainable in this way.*

ii) *Solving an n-th order ODE is* **likely** *to involve n integrations. Thus, the general solution of an n-th order ODE involves n arbitrary constants.*

Definition

First order separable ODEs can be written in the form

$$\frac{dy}{dx} = g(x)h(y)$$

We can rewrite this as $\frac{1}{h(y)}\frac{dy}{dx} = g(x)$.

Example (Solution of some ODEs)

i) a) Find the GS of $\frac{dy}{dx} = 4x^3y^2$.

   b) Find the PS satisfying $y(0) = 1$.

ii) Solve $\frac{dy}{dx} = (6x^2 + 1)y$, where y is a positive variable.

iii) Find the GS of $\frac{dy}{dx} = ky$ where k is constant.

Definition (Integrating factor)

An **integrating factor (IF)** $\mu(x)$ has the property that multiplication of the expression

$$p(x)\frac{dy}{dx} + q(x)y(x)$$

by $\mu(x)$ turns the expression into a total derivative

$$\frac{d}{dx}\left(\mu(x)p(x)y(x)\right).$$

Example

Solve the following first order ordinary differential equations by finding an integrating factor.

i) $x\dfrac{dy}{dx} - 3y = x^5$

ii) $\dfrac{dy}{dx} + \left(\dfrac{x+2}{x}\right)y = e^{-x}$

2.3 Second order linear ODEs with constant coefficients

Definition

Second order linear ODEs with constant coefficients take the form

$$py'' + qy' + ry = f(x), \tag{1}$$

where p, q, r are constants, $y(x)$ is the solution to the ODE and $f(x)$ is some function. When $f(x) = 0$ for all x then (1) is a *homogeneous equation*. When $f(x) \neq 0$ for some x then (1) is an *inhomogeneous equation*.

Lemma

Suppose $y = \varphi(x)$ *is a solution to* (1)*, then the general solution of equation* (1) *is*

$$y(x) = CF + PI = CF + \varphi(x),$$

where the complementary function (CF) is the general solution to $py'' + qy' + ry = 0$.

2.3 Second order linear ODEs with constant coefficients

Theorem (General solution of homogeneous second-order ODE)

The exponential e^{mx} is a solution of $py'' + qy' + ry = 0$ if m satisfies the **Auxiliary equation** *(AE)*

$$pm^2 + qm + r = 0.$$

Technique

AE has	*general solution,* $y(x) =$
2 distinct real roots, m_1, m_2	$c_1e^{m_1x} + c_2e^{m_2x}$
2 repeated roots, m	$c_1e^{mx} + c_2xe^{mx}$
2 complex roots, $\alpha \pm i\beta$	$e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$

Example (Find the general solutions to the following ODEs)

i) $y'' - 5y' + 6y = 0$

ii) $y'' - 6y' + 9y = 0$

iii) $y'' - 2y + 5y = 0$

iv) $y'' + 9y = 0$

v) $y'' - 9y = 0$

A special 2nd order ODE with constant coefficients is:

$$\frac{d^2y}{dx^2} + \omega^2 y = 0, \tag{2}$$

where $\omega > 0$ is a constant.

This equation describes small oscillations of a pendulum or of a mass attached to a spring obeying Hooke's law (force $\sim$ compression).
The general solution of (2) is

$$y = c_1 \cos \omega x + c_2 \sin \omega x. \tag{3}$$

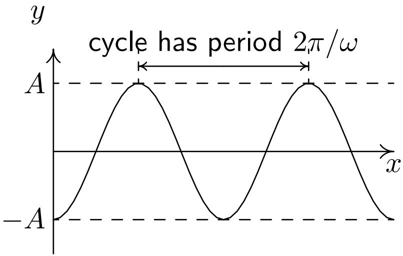
where c_1 and c_2 are arbitrary constants.
(3) can be alternatively expressed in **amplitude–phase** form as

$$y = A \sin(\omega x + \varphi), \tag{4}$$

where A and φ are an arbitrary constants.

2.4 Simple harmonic motion

Solutions, $y = A\sin(\omega x + \varphi)$, of $y'' + \omega^2 y = 0$ are oscillations, so called **simple harmonic oscillations**.



- Every nonzero solution of $y'' + \omega^2 y = 0$ is a **simple harmonic oscillator** with **period** $2\pi/\omega$.
- The frequency of oscillations is ω (in radians per second) or $\omega/2\pi$ (in cycles per second or Hz).
- The constant A is the **amplitude** of the oscillation and φ is the **phase**.

2.4 Simple harmonic motion with damping

Most physical oscillatiors are often subject to small frictional/resistant force proportional to the velocity $\frac{dy}{dx}$. Then the governing ODE for damped simple harmonic motion, with coefficient of damping k is,

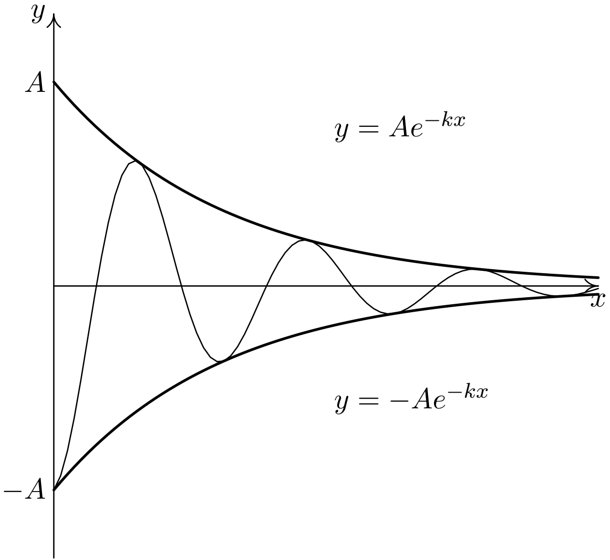
$$y'' + \underbrace{2ky'}_{\text{friction}} + \omega^2 y = 0. \quad (5)$$

coefficient of damping, k	system is
$k < \omega$	underdamped
$k > \omega$	overdamped
$k = \omega$	critically damped

The general solution of (5) is

$$y = e^{-kx}(c_1 \cos\sqrt{\omega^2 - k^2}x + c_2 \sin\sqrt{\omega^2 - k^2}x),$$

or equivalently $y = Ae^{-kx}\sin(\sqrt{\omega^2 - k^2}x + \alpha)$.
This is **damped oscillation**.



The envelope of oscillations is Ae^{-kx}

2.5 Finding particular integrals (PIs)

Particular integrals are particular solutions $y(x)$ to the inhomogenous

$$py'' + qy' + ry = f(x).$$

Technique

$f(x)$	*trial PI*, $y(x)$	*conditions of AE*
polyn. of deg. n	$a_0 + a_1 x + \cdots + a_n x^n$	
$a \cos \alpha x + b \sin \alpha x$	$A \cos \alpha x + B \sin \alpha x$	$i\alpha$ *not a root*
	$x(A \cos \alpha x + B \sin \alpha x)$	$i\alpha$ *is a root*
$ae^{\alpha x}$	$Ae^{\alpha x}$	α *not a root*
	$Axe^{\alpha x}$	α *is a non-rep. root*
	$Ax^2 e^{\alpha x}$	α *is a repeated root*

Note If $f(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, where each $f_i(x)$ is one of the types above, then find PIs $y_1(x), y_2(x), \ldots, y_n(x)$ for each $f_i(x)$ separately. Then the PI is $y(x) = y_1(x) + y_2(x) + \cdots + y_n(x)$.

2.5 Finding particular integrals (PIs)

Example (Find the general solution for the following ODEs)

i) Find the General Solution of $y'' + 3y' + 2y = 6e^{2x}$.

ii) Find the General Solution of $y'' + 3y' + 2y = 10\cos 2x$.

Example (Extra examples)

Find the general solution to the following inhomogeneous ODEs:

i) $y'' - 2y' - 8y = 10e^{-x}$.

ii) $y'' - 2y' - 8y = 12e^{4x}$.

iii) $y'' - 4y' + 4y = 6e^{2x}$.

iv) $y'' - 6y' + 5y = 6\cos x + 22\sin x$.

v) $y'' + 9y = -24\cos 3x - 6\sin 3x$.

vi) $y'' + 8y' + 16y = 16x^2 + 26$.