# A Glance Through the Gate

## Clifford's Geometrical Algebra



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*..for geometry, you know, is the*
*gate of science, and the gate is*
*so low and small that one*
*can only enter it as a little child.*

*– William Kingdom Clifford*

# Preface

Clifford's algebras are a rich subject, born in the later half of the nineteenth century, they have been lying dormant for many years. Recently they have received much interest from both physicists and computer scientists; this in turn has generated new interest in their mathematics. Presented here is an introduction to this diverse subject. It is aimed at a final year undergraduate or first year postgraduate interested in discovering the basic notions and ideas concerned with this area of mathematics.

We assume, for the purpose of this report, that the reader is comfortable with basic vector space, matrix and ring theory. In particular with; basic set theory, basis and dimension of vector spaces, divisors of zero, direct products/sums of rings and vector spaces, homomorphisms, surjections, injections, bijections, the kernel and image of homomorphisms, ideals, generating sets, the orthogonal group and special orthogonal group.

Perhaps it is worthy of note that in many places this report is not as general as could be the case. Preference throughout has been given to presenting the underlying concepts as clearly as possible. This approach will hopefully impart to the reader an intuitive feel for algebras. As an appendix we have included a more general definition of a Clifford algebra for the interested reader.

Section 1 presents a basic introduction to algebras. We assume that the reader is familiar with some basic examples of vector spaces and rings and use these as our starting point. As is often the case with new mathematics, the ratio of definitions to theorems is rather high in this section.

Section 2 gives some of the reasons for the importance of matrix algebras in studying algebras in general, including a method for finding the regular representations of any given algebra.

Section 3 contains the main exposition of this report, here we acquaint the reader with Clifford algebras and guide them through the basic definitions. The section continues with a discovery of the regular representations of these algebras and examines a fascinating theorem describing the 'periodicity' of Clifford algebras.

Section 4 is included to give the reader an idea of which directions further study could lead. Also offered here are some applications for this theory to other sciences.

I would especially like to thank Andy Baker for his guidance in helping me to understand the material in this report.

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Cover Illustration: Dave Chisholm's depiction of William Clifford performing his 'corkscrew'. It appears that Clifford was not only theoretically but also athletically expert at rotations!

Figure 7: an illustration by Roger Penrose from his book [Pnrse 2004].

# Introduction

Confusingly, mathematicians use the word ‘algebra’ in two distinct ways. On the one hand algebra is a broad branch on the mathematical tree of knowledge; on the other, it is the name given to a very precise structure. It is often the case in mathematics that where subject areas overlap, new fertile and fascinating areas arise from their intersection. This is the case with algebras, where vectors spaces and rings interact together in intriguing ways.

In this report we shall take the approach of expanding a vector space in such a way as to allow multiplication of vectors.

What shall we require of this product? It would be advantageous to have it satisfy the same axioms as the multiplication for real numbers, that is distributivity, associativity and commutivity. How about the complex numbers? Vectors in $\mathbb{R}^2$ can be represented by complex numbers, whose addition and multiplication satisfies the same axioms as the real numbers. Can this be extended for dimensions greater than two?

It turns out that there is no way to develop these axioms for larger dimensions. In 1843, considering a similar question was William Hamilton. Whilst walking along the Royal Canal in Dublin with his wife, he had a flash of insight. In an infamous act he defaced Brougham bridge with the quaternions defining equations

$$i^2 = j^2 = k^2 = -1$$

succeeding in generalising the concept of the complex numbers to four dimensions. He achieved this feat by dropping the requirement of commutivity.

Thirty years later, another William, William Kingdom Clifford desired a consistent framework for extending these observations to higher dimensions. This is what we study in detail here: *Clifford’s algebras*.

On the path to investigating Clifford’s algebras we shall encounter rich theory, including matrix algebras and their importance in understanding algebras in general. Finally, we conclude the report by pointing the reader to future paths for study and research. In addition to this, we unearth applications for Clifford algebras – revealing , amongst other things, their significance in modern physics.

# 1. Algebras

Informally, an algebra is a extension of the structure of a vector space. In addition to the vector space axioms, we define a multiplication between vectors in such a way as to form a ring.

Let $\mathbb{F}$ be a field throughout.

More formally then, we have the following.

**Definition 1.** An *algebra* $\mathcal{A}$ over $\mathbb{F}$ is a set on which three operations are defined; addition, multiplication and multiplication by scalars. These must satisfy the following conditions.

(i) $\mathcal{A}$ is a ring under addition and multiplication.
(ii) $\mathcal{A}$ is an $\mathbb{F}$-vector space under addition and multiplication by scalars.
(iii) $\forall a,b \in \mathcal{A}$ and $\lambda \in \mathbb{F}$

$$(\lambda a)b = a(\lambda b) = \lambda(ab).$$

Condition (iii) ensures that scalar multiplication commutes among the vectors in a sensible way.

Since an algebra $\mathcal{A}$ is both a vector space and a ring, we use definitions and terminology established in connection with these structures to describe $\mathcal{A}$. For example, we may refer to elements of $\mathcal{A}$ as vectors; we may consider bases for $\mathcal{A}$; we may also consider what it means for $\mathcal{A}$ to have divisors of zero and so on. It is in this spirit we make the following definition.

**Definition 2.** An algebra $\mathcal{A}$ is said to be of *dimension n* when it is $n$ dimensional as an $\mathbb{F}$-vector space. When $\mathcal{A}$ is an infinite dimensional $\mathbb{F}$-vector space, we say that it is of *infinite dimension* as an algebra.

**Examples 3.** (i) As a first example, the familiar complex numbers $\mathbb{C}$ can be thought of as an $\mathbb{R}$-algebra of dimension 2, with basis $\{1,i\}$. Let's work through this in detail.
Any elements $\alpha,\beta \in \mathbb{C}$ can be written as

$$\alpha = a_1 1 + a_2 i \qquad \beta = b_1 1 + b_2 i$$

with $a_1,a_2,b_1,b_2 \in \mathbb{R}$. Multiplication is then,

$$\alpha\beta = (a_1 1 + a_2 i)(b_1 1 + b_2 i) = a_1 b_1 1 + (a_1 b_2 + a_2 b_1)i + a_2 b_2 i^2$$

but of course $i^2 = -1$, so we have,

$$\alpha\beta = (a_1 b_1 - a_2 b_2)1 + (a_1 b_2 + a_2 b_1)i.$$

The multiplication of vectors in an algebra can be described completely in terms of multiplication of basis elements. Indeed, every algebra has a unique multiplication table, with respect to a particular basis. Below in Table 1 we see the multiplication table for $\mathbb{C}$, with this basis.

| | 1 | $i$ |
|---|---|---|
| 1 | 1 | $i$ |
| $i$ | $i$ | $-1$ |

Table 1. Multiplication Table for $\mathbb{C}$

(ii) Next, the quaternions $\mathbb{H}$, of which we will see a lot more later, are a 4-dim $\mathbb{R}$-algebra, with standard basis $\{1, i, j, k\}$, whose multiplication table is shown in Table 2.

| | 1 | $i$ | $j$ | $k$ |
|---|---|---|---|---|
| 1 | 1 | $i$ | $j$ | $k$ |
| $i$ | $i$ | $-1$ | $k$ | $-j$ |
| $j$ | $j$ | $-k$ | $-1$ | $i$ |
| $k$ | $k$ | $j$ | $-i$ | $-1$ |

Table 2. Multiplication Table for $\mathbb{H}$

Notice that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$.

(iii) What about $\mathbb{H}$ as a $\mathbb{C}$-algebra? On inspection, we see that $\mathbb{H}$ is both a ring and a $\mathbb{C}$-vector space. Unfortunately, the scalars (i.e. elements of $\mathbb{C}$) do not commute evenly among the vectors. For example, take $\lambda = i, a = j$ and $b = 1$ in condition (iii) of the definition to get

$$(ij)1 = j(i1) = i(j1) \qquad \text{i.e. } ij = ji$$

which, on checking Table 2, is clearly false.

(iv) $\mathcal{M}_n(\mathbb{F})$, the $n \times n$ matrices over $\mathbb{F}$ are an $n$ dimensional $\mathbb{F}$-algebra, these are very important examples of algebras. For this reason we devote Section 2 to studying their basic properties.

(v) Let $G$ be a group. Then we may take the group elements as a basis of an $\mathbb{F}$-vector space and form what is called the *group algebra* over $\mathbb{F}$, sometimes denoted $\mathbb{F}G$. The group multiplication table then becomes a multiplication table for the algebra. Although we won't go into any more detail here on this subject, it is worth noting that these algebras are very important in studying groups.

**Note 4.** As with finite groups, to show an isomorphism between algebras it is enough to show that their multiplication tables are the same.

We conclude this introduction to algebras by making some definitions that will be useful in the following sections. These should appear familiar to the reader; they are direct analogues of similar definitions made in group, ring or module theory.

For the remainder of this section, let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be algebras over the same field $\mathbb{F}$.

**Definition 5.** We say that a $\mathcal{B}$ is a *sub-algebra* of $\mathcal{A}$, if $\mathcal{B}$ is an algebra under the same operations as $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}$.

**Example 6.** For example, $\mathbb{C}$ is a sub-algebra of $\mathbb{H}$ (as $\mathbb{R}$-algebras). To prove this is it enough to show that the multiplication table of $\mathbb{C}$ is contained in the multiplication table for $\mathbb{H}$, which can be easily checked in Tables 1 and 2. In fact, the multiplication table for $\mathbb{C}$ is contained more than once. That is to say, there is more than one subalgebra of $\mathbb{H}$ that is isomorphic to $\mathbb{C}$, as an algebra.

**Definition 7.** The *direct product* of $\mathcal{B}$ and $\mathcal{C}$ is defined to be the set $\mathcal{B} \times \mathcal{C} = \{(b,c) : b \in \mathcal{B}, c \in \mathcal{C}\}$ satisfying, for all $b, b' \in \mathcal{B}, c, c' \in \mathcal{C}$ and $\lambda \in \mathbb{F}$.

(i) $(b,c) + (b',c') = (b+b', c+c')$
(ii) $(b,c)(b',c') = (bb', cc')$
(iii) $\lambda(b,c) = (\lambda b, \lambda c)$

Note that these conditions imply the following.

$$\big(\lambda(b,c)\big)(b',c') = (b,c)\big(\lambda(b',c')\big) = \lambda\big((b,c)(b',c')\big)$$

Clearly this definition can be extended, by induction, to direct products of any number of algebras.

**Definition 8.** A map $\varphi : \mathcal{B} \to \mathcal{C}$ is an *algebra homomorphism* if it is both a ring and a vector space homomorphism. That is for all $b, b' \in \mathcal{B}$ and $\lambda \in \mathbb{F}$, $\varphi$ satisfies the following.

(i) $\varphi(b+b') = \varphi(b) + \varphi(b')$
(ii) $\varphi(bb') = \varphi(b)\varphi(b')$
(iii) $\varphi(\lambda b) = \lambda\varphi(b)$

We say that $\varphi$ is a *monomorphism* (respectively *epimorphism*) when it is injective (respectively surjective). When $\varphi$ is both a monomorphism and an epimorphism we say that it is an *isomorphism*, writing $\mathcal{B} \cong \mathcal{C}$.

**Definition 9.** Let $\varphi : \mathcal{B} \to \mathcal{C}$ be an algebra homomorphism. Then we define the *kernel* of $\varphi$, written $\ker\varphi$, to be the set

$$\ker\varphi = \{b \in \mathcal{B} : \varphi(b) = 0\}.$$

We also define the *image* of $\varphi$, written $\mathrm{im}\varphi$, to be the set

$$\mathrm{im}\varphi = \{c \in \mathcal{C} : c = \varphi(b), \text{ for some } b \in \mathcal{B}\}.$$

We should perhaps note that, by construction, any algebra homomorphism is simultaneously a vector space and ring homomorphism, for which we already have definitions of the image and kernel. It may be easily checked however that these definitions coincide.

**Theorem 10.** ($1^{\text{st}}$ Isomorphism Theorem for Algebras)
*Let $\varphi : \mathcal{B} \to \mathcal{C}$ be an algebra homomorphism. Then $\ker\varphi$ is an ideal of $\mathcal{B}$, $\mathrm{im}\varphi$ is a subalgebra of $\mathcal{C}$ and*

$$\mathcal{B}/\ker\varphi \cong \mathrm{im}\varphi.$$

This is an important theorem, essentially commenting on the possible factorisations of any given homomorphism. Perhaps it is more readily absorbed in a pictorial form, displayed below.

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathcal{C} \\
\mathcal{B}/\ker\varphi \arrow[r, "\widetilde{\varphi}"] & \mathrm{im}\varphi \arrow[u, "\iota"']
\end{tikzcd}$$

Here $\pi : \mathcal{B} \to \mathcal{B}/\ker\varphi$ is the projection map sending $b \mapsto b + \ker\varphi$, $\widetilde{\varphi}$ is the isomorphism of the theorem and $\iota : \mathrm{im}\varphi \to \mathcal{C}$ is the inclusion map $c \mapsto c$. The theorem can then be stated by saying that the above diagram commutes, that is $\varphi = \iota \circ \widetilde{\varphi} \circ \pi$.

In other algebraic structures, groups, modules and rings for instance, analogues of the first isomorphism theorem are a powerful tool in identifying isomorphisms between spaces. This power is increased in algebras since we can bring to bear on problems dimension arguments. For example, with $\varphi$ as above, if we know that $\dim(\mathrm{im}\varphi) = \dim\mathcal{C}$ then as $\mathrm{im}\varphi$ is a subalgebra $\mathcal{C}$ we may conclude that $\mathrm{im}\varphi = \mathcal{C}$.

**Exercises.** Let $\varphi : \mathcal{B} \to \mathcal{C}$ be an algebra homomorphism.

(i) Show that the direct product of two algebras is also an algebra.

(ii) Show that $\ker\varphi$ is an ideal of $\mathcal{B}$ and $\mathrm{im}\varphi$ is a sub-algebra of $\mathcal{C}$.

(iii) Prove that $\quad \varphi$ is injective $\iff \ker\varphi = 0$.

(iv) Prove also that $\quad \varphi$ is surjective $\iff \mathrm{im}\varphi = \mathcal{C}$.

(v) Find the isomorphism of Theorem 10.

# 2. Matrix Algebras

We noted in the previous section that matrix algebras are very important in algebra theory. Without hesitation then, we present the reason for this statement.

**Theorem 11.** *Every algebra $\mathcal{A}$ over $\mathbb{F}$ is isomorphic to a subalgebra of the matrix algebra $\mathcal{M}_n(\mathbb{F})$, for some n.*

**Proof** (Abian; [Abn 1971])
We will prove the theorem for the case $n = 3$, which illustrates the proof in general. Thus, we assume that $\mathcal{A}$ is a 3 dimensional algebra over $\mathbb{F}$ and that $\{I, A, B\}$ is a basis of $\mathcal{A}$ where $I$ is the identity of $\mathcal{A}$. Let the multiplication table of $\mathcal{A}$ with respect to the basis $\{I, A, B\}$ be given by

| | $I$ | $A$ | $B$ |
|---|---|---|---|
| $I$ | $1I+0A+0B$ | $0I+1A+0B$ | $0I+0A+1B$ |
| $A$ | $0I+1A+0B$ | $aI+bA+cB$ | $gI+hA+kB$ |
| $B$ | $0I+0A+1B$ | $qI+rA+tB$ | $uI+vA+wB$ |

Clearly, we may represent $I$ by $(1,0,0)$, $A$ by $(0,1,0)$ and $B$ by $(0,0,1)$ and rewrite this table as follows.

| | $I$ | $A$ | $B$ |
|---|---|---|---|
| $(1,0,0)$ | $(1,0,0)$ | $(0,1,0)$ | $(0,0,1)$ |
| $(0,1,0)$ | $(0,1,0)$ | $(a,b,c)$ | $(g,h,k)$ |
| $(0,0,1)$ | $(0,0,1)$ | $(q,r,t)$ | $(u,v,w)$ |

From the rewritten table we see that

$$(1,0,0).I = (1,0,0); \quad (0,1,0).I = (0,1,0); \quad (0,0,1).I = (0,0,1).$$

The above equalities suggest substituting

$$\text{for } I \text{ the matrix } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, from this table we see that

$$(1,0,0).A = (0,1,0); \quad (0,1,0).A = (a,b,c); \quad (0,0,1).A = (q,r,t).$$

The above equalities suggest substituting

$$\text{for } A \text{ the matrix } \begin{pmatrix} 0 & 1 & 0 \\ a & b & c \\ q & r & t \end{pmatrix}.$$

Finally, we see that

$$(1,0,0).B = (0,0,1); \quad (0,1,0).B = (g,h,k); \quad (0,0,1).B = (u,v,w).$$

The above equalities suggest substituting

$$\text{for } B \text{ the matrix } \begin{pmatrix} 0 & 0 & 1 \\ g & h & k \\ u & v & w \end{pmatrix}.$$

Motivated by these substitutions, we consider the mapping $\varphi$ from $\mathcal{A}$ into $\mathcal{M}_3(\mathbb{F})$ where for every element $S \in \mathcal{A}$ with $S = s_1 I + s_2 A + s_1 B$ we define

$$\varphi(S) = s_1 \varphi(I) + s_2 \varphi(A) + s_1 \varphi(B)$$

where

$$\varphi(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \varphi(A) = \begin{pmatrix} 0 & 1 & 0 \\ a & b & c \\ q & r & t \end{pmatrix} \quad \varphi(B) = \begin{pmatrix} 0 & 0 & 1 \\ g & h & k \\ u & v & w \end{pmatrix}.$$

We claim that $\varphi$ is an algebra homomorphism and leave it as an exercise to check the three conditions prescribed in Definition 8 hold.

Finally, we see the matrices $\varphi(I), \varphi(A)$ and $\varphi(B)$ are linearly independent. Therefore $\ker \varphi = 0$. By the first isomorphism theorem (Theorem 10), we have that $\varphi$ is an isomorphism from $\mathcal{A}$ onto a subalgebra of $\mathcal{M}_3(\mathbb{F})$, as desired. $\square$

This process of representing an algebra over $\mathbb{F}$ by a subalgebra of the matrix algebra is sometimes referred to as *regular representation*.

**Example 12.** For a concrete example of Theorem 11 we shall endeavor to discover the regular representation of the algebra $\mathbb{C}$. First we rewrite Table 1 as

| | $1$ | $i$ |
|---|---|---|
| $(1,0)$ | $(1,0)$ | $(0,1)$ |
| $(0,1)$ | $(0,1)$ | $(-1,0)$ |

which suggests the mappings

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad i \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and a quick check

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

shows that we are on the right track.
Finally we see that,

$$\mathbb{C} \cong \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

**Exercise.** Find the regular representation for $\mathbb{H}$, the quaternions.

Another reason for matrix algebras importance is the following.

**Proposition 13.** *Let $\mathcal{A}$ be an algebra with $\mathcal{B}$ and $\mathcal{M}_n(\mathbb{F})$ as subalgebras such that*

$$\mathcal{B} \cap \mathcal{M}_n(\mathbb{F}) = \{\lambda.1_{\mathcal{A}} : \lambda \in \mathbb{F}\}.$$

*Suppose that $\mathcal{B}$ and $\mathcal{M}_n(\mathbb{F})$ commute element-wise and that $\mathcal{B}$ and $\mathcal{M}_n(\mathbb{F})$ generate $\mathcal{A}$. Then*

$$\mathcal{A} \cong \mathcal{M}_n(\mathcal{B}).$$

This is, for those readers who are familiar, ostensibly a fact from tensor products (i.e. $\mathcal{B} \otimes \mathcal{M}_n(\mathbb{F}) \cong \mathcal{M}_n(\mathcal{B})$). Indeed this is the correct way to think of this result. However, since this is not the place to introduce such theory we will give an illuminating example in place of a formal proof.

**Example 14.** Consider the matrix algebra $\mathcal{M}_2(\mathbb{C})$. This has subalgebras $\mathbb{C}$ and $\mathcal{M}_2(\mathbb{R})$, with bases

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right\} \qquad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

respectively.

It is easy to see that elements of these bases commute with each other and that

$$\mathbb{C} \cap \mathcal{M}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}.$$

Further, they generate an algebra with basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \right\}$$

i.e. with basis

$$\left\{ \begin{pmatrix} 1+i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1+i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1+i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1+i \end{pmatrix} \right\}$$

Which can be seen to be a basis for $\mathcal{M}_2(\mathbb{C})$.

# 3. Clifford Algebras

With our introduction to algebras finished, what follows is the main exposition of this report. Starting with the basic definition below, we shall expand to look at the regular representations of Clifford algebras. Culminating this section with a marvellous theorem (Theorem 22) on the 'periodicity' of Clifford algebras.

**Definition 15.** Given a real vector space $V$, the *Clifford algebra* $C\ell_{\mathrm{V}}$ is the associative algebra freely generated by $V$ satisfying

$$x^2 = -|x|^2$$

for all $x \in V$.

We restrict our exploration of Clifford algebras here to considering only the case where

$$V = \mathbb{R}^n \qquad \text{and} \qquad |x| = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}$$

where $x = (x_1, x_2, \ldots, x_n)$, we denote this family of Clifford algebras $C\ell_{\mathrm{n}}$. A more general definition is included in the appendix as Definition 24. For a broader approach to Clifford algebras see [Artn 1957], [Bkr 2002], [Lousto 1995] and [Prts 1995].

**Example 16.** Let's get our hands dirty with some calculations.
Take an orthonormal basis of $\mathbb{R}^2$, $\{e_1, e_2\}$. We wish to form all finite products of these basis elements, subject to the condition given in the definition. Any vector in $\mathbb{R}^2$ can be written as $\lambda e_1 + \mu e_2$ for some $\lambda, \mu \in \mathbb{R}$. Using the definition above we require that

$$\begin{aligned} (\lambda e_1 + \mu e_2)^2 &= -(\lambda^2 + \mu^2), \\ \text{i.e.} \quad \lambda^2 e_1^2 + \mu^2 e_2^2 + \lambda\mu(e_1 e_2 + e_2 e_1) &= -\lambda^2 - \mu^2 \end{aligned}$$

Hence, equating coefficients, we have the relations

$$e_1^2 = e_2^2 = -1 \quad \text{and} \quad e_1 e_2 + e_2 e_1 = 0\,.$$

Thus any further products of basis elements is linearly dependant on the elements $\{1, e_1, e_2, e_1 e_2\}$ (where 1 denotes the empty product). Therefore, this set forms a basis for $C\ell_2$.

Observe that the subset of this basis $\{e_1, e_2\}$ of products of odd length generates $\mathbb{R}^2$ as a vector space. Also that the products of even length are closed under multiplication showing that the space spanned by $\{1, e_1 e_2\}$ is a subalgebra of $C\ell_2$.

The full multiplication table for $C\ell_2$ is shown below in Table 3 below.

| | $1$ | $e_1$ | $e_2$ | $e_1e_2$ |
|---|---|---|---|---|
| $1$ | $1$ | $e_1$ | $e_2$ | $e_1e_2$ |
| $e_1$ | $e_1$ | $-1$ | $e_1e_2$ | $-e_2$ |
| $e_2$ | $e_2$ | $-e_1e_2$ | $-1$ | $e_1$ |
| $e_1e_2$ | $e_1e_2$ | $e_2$ | $-e_1$ | $-1$ |

Table 3. Multiplication Table for $C\ell_2$

Notice the similarity between this multiplication table and Table 2 for the quaternions. As we remarked in Note 4, this is sufficient to show the following.

**Proposition 17.**

$$C\ell_2 \cong \mathbb{H}$$

As a corollary to this, we note that the even subalgebra mentioned above, spanned by $\{1, e_1e_2\}$ is isomorphic to the complex numbers $\mathbb{C}$.

Generalising the process followed in the example, by building $C\ell_{\mathrm{n}}$ from an orthonormal basis of $\mathbb{R}^n$, we recover the subsequent proposition.

**Proposition 18.** *For $1 \leq i \leq n$, let $\{e_i\}$ be an orthonormal basis for $\mathbb{R}^n$. Then in $C\ell_{\mathrm{n}}$ we have the following relations.*

$$e_i^2 = -1 \qquad e_ie_j + e_je_i = 0$$

*for $1 \leq i, j \leq n$ with $i \neq j$.*

**Remarks.** In our definition of Clifford algebras, Definition 15, we made two assertions that are not at all transparent. Firstly, we said that a Clifford algebra is *freely* generated by the vector space; secondly, that this algebra is unique (up to isomorphism).

The concern is that we have made choices that could apparently lead to the construction of distinct Clifford algebras. Namely, we choose a basis of the vector space $V$, then we multiply elements of this basis together to get the basis for an algebra. The miraculous part is if we choose a different basis of $V$ and form the Clifford algebra over $V$ with respect to this new basis, we get an isomorphic Clifford algebra! For more details on this consult [Bkr 2002] or [Chvy 1997]. An algebra with this property is often called an *universal* algebra.

### 3.1. **Grade and Dimension.**

**Definition 19.** We define the *grade* of a basis element of $C\ell_{\mathrm{n}}$ to be the length of the product of that element.

For example, the basis of $C\ell_2$ consists of:

| | | | |
|---|---|---|---|
| 1 | element of grade zero | $1$ | scalar |
| 2 | elements of grade one | $e_1, e_2$ | vectors |
| 1 | element of grade two | $e_1e_2$ | bi-vector |

In general, the basis of $C\ell_n$ consists of:

| | | | |
|---|---|---|---|
| 1 | element of grade zero | $1$ | scalar |
| $n$ | elements of grade one | $e_1, \ldots, e_n$ | vectors |
| $\binom{n}{2}$ | elements of grade two | $e_1e_2, \ldots, e_{n-1}e_n$ | bi-vectors |
| $\binom{n}{3}$ | elements of grade three | $e_1e_2e_3, \ldots, e_{n-2}e_{n-1}e_n$ | tri-vectors |
| | $\vdots$ | | $\vdots$ |
| 1 | element of grade $n$ | $e_1e_2\cdots e_n$ | $n$-vector |

The grade structure of Clifford algebras follows the pattern of Pascal's triangle, seen below.

| **n** | | | | | | | | | | | | | | | | | **$2^n$** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | 1 | | | | | | | | | 1 |
| 1 | | | | | | | 1 | | 1 | | | | | | | | 2 |
| 2 | | | | | | 1 | | 2 | | 1 | | | | | | | 4 |
| 3 | | | | | 1 | | 3 | | 3 | | 1 | | | | | | 8 |
| 4 | | | | 1 | | 4 | | 6 | | 4 | | 1 | | | | | 16 |
| 5 | | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | | | | 32 |
| 6 | | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 | | | 64 |
| 7 | 1 | | 7 | | 21 | | 35 | | 35 | | 21 | | 7 | | 1 | | 128 |
| $\vdots$ | | $\vdots$ | | $\vdots$ | | $\vdots$ | | $\vdots$ | | $\vdots$ | | $\vdots$ | | $\vdots$ | | | $\vdots$ |

From vector spaces, we know that its dimension is the number of elements in its basis. Thus the dimension of an algebra is the number of elements in the basis. The above remarks on the grade structure of Clifford algebras give us the following proposition. Indeed, this also follows quite naturally from the fact that we are constructing our algebras freely.

**Proposition 20.**

$$\dim C\ell_n = 2^n.$$

3.2. **Some Remarks.**

(i) We have throughout been constructing examples of Clifford algebras from the vector space $\mathbb{R}^n$ with the standard inner product. Reflecting on the containment

$$1 \subset \mathbb{R} \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n \subset \cdots$$

it follows that

$$C\ell_0 \subset C\ell_1 \subset C\ell_2 \subset \cdots \subset C\ell_{\mathrm{n}-1} \subset C\ell_{\mathrm{n}} \subset \cdots$$

(ii) Note also that in $C\ell_3$,

$$(1+e_1e_2e_3)(1-e_1e_2e_3) = 0,$$

with $(1+e_1e_2e_3), (1-e_1e_2e_3) \neq 0$.

Thus $C\ell_3$ has a divisor of zero.

These two remarks bring to light the following.

**Proposition 21.** *For $n > 2$, $C\ell_{\mathrm{n}}$ has divisors of zero.*

3.3. **Matrix Representations of Clifford Algebras.** In the spirit of the previous section (Section 2) on matrix algebras, we show here isomorphisms between the first nine Clifford algebras and standard matrix algebras. As we will see in Section 3.4, something quite wonderful happens after then.

We saw in Example 16 that $C\ell_2 \cong \mathbb{H}$ and we hinted in the introduction that there exist Clifford algebras isomorphic to $\mathbb{R}$ and $\mathbb{C}$.

A method due to Alan Wiederhold gives us a uniform way of obtaining the representations of $C\ell_0$ to $C\ell_4$ shown below in table 4.

| $n$ | $C\ell_{\mathrm{n}}$ |
|---|---|
| 0 | $\mathbb{R}$ |
| 1 | $\mathbb{C}$ |
| 2 | $\mathbb{H}$ |
| 3 | $\mathbb{H} \times \mathbb{H}$ |
| 4 | $\mathcal{M}_2(\mathbb{H})$ |

Table 4. Matrix Representations of $C\ell_0$ to $C\ell_4$

From our knowledge of matrix theory, we know that any element $\alpha$ of a field $\mathbb{F}$ can be represented with the element $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \in \mathcal{M}_2(\mathbb{F})$ . Further, we can represent an element $(\alpha_1, \alpha_2)$ of $\mathbb{F} \times \mathbb{F}$ with the element $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \in \mathcal{M}_2(\mathbb{F})$ .

Using this knowledge, combined with the observations

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{H} \times \mathbb{H} \subset \mathcal{M}_2(\mathbb{H})$$

$$C\ell_0 \subset C\ell_1 \subset C\ell_2 \subset \cdots \subset C\ell_{n-1} \subset C\ell_n \subset \cdots$$

we consider the following assignments.

$$\begin{tikzcd}
 & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \\
\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} & & 1 \arrow[u, mapsto] & & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \\
 & e_4 \arrow[ul, mapsto] & & e_1 \arrow[ur, mapsto] & \\
 & e_3 \arrow[dl, mapsto] & & e_2 \arrow[dr, mapsto] & \\
\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} & & & & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}
\end{tikzcd}$$

where $i$, $j$ and $k$ are defined as on Brougham bridge! That is $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$.

The dedicated reader may check that this indeed gives the isomorphisms in Table 4.

Next we would like to find a representation for $C\ell_5$. The assignment of the basis vectors $1, e_1, \ldots, e_5$ of $\mathbb{R}^5$ shown below induces the required isomorphism $C\ell_5 \to \mathcal{M}_4(\mathbb{C})$. Here, as usual, $i^2 = -1$.

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e_1 \mapsto \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \qquad e_5 \mapsto \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$e_2 \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad e_4 \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$e_3 \mapsto \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

By direct calculation, using the method described in the proof of Theorem 11 for finding a regular representation, we discover the representations shown below in Table 5.

| $n$ | $C\ell_{\mathrm{n}}$ |
|---|---|
| 6 | $\mathcal{M}_8(\mathbb{R})$ |
| 7 | $\mathcal{M}_8(\mathbb{R}) \times \mathcal{M}_8(\mathbb{R})$ |
| 8 | $\mathcal{M}_{16}(\mathbb{R})$ |

Table 5. Matrix Representations of $C\ell_6$ to $C\ell_8$

3.4. **Periodicity of 8.** The result that was alluded to at the start of the last section is the topic of this section. Informally, it characterises all of the real Clifford algebras (with the standard inner product) in terms of the first eight, which we have summerised below in Table 6.

| $n$ | $C\ell_n$ |
|---|---|
| 0 | $\mathbb{R}$ |
| 1 | $\mathbb{C}$ |
| 2 | $\mathbb{H}$ |
| 3 | $\mathbb{H} \times \mathbb{H}$ |
| 4 | $\mathcal{M}_2(\mathbb{H})$ |
| 5 | $\mathcal{M}_4(\mathbb{C})$ |
| 6 | $\mathcal{M}_8(\mathbb{R})$ |
| 7 | $\mathcal{M}_8(\mathbb{R}) \times \mathcal{M}_8(\mathbb{R})$ |

Table 6. Matrix Representations of $C\ell_0$ to $C\ell_7$

**Theorem 22.** (Cartan, 1908)

$$C\ell_{n+8} \cong \mathcal{M}_{16}(C\ell_n)$$

**Proof**

Take an orthonormal basis $\{e_1, e_2, \dots, e_n, e_{n+1}, \dots, e_{n+8}\}$ of $\mathbb{R}^{n+8}$ and set $e'_i = e_i e_{n+1} \cdots e_{n+8}$ for $i = 1, 2, \dots, n$. Then the subset $\{e'_1, e'_2, \dots, e'_n\}$ of $C\ell_{n+8}$ generates a subalgebra isomorphic to $C\ell_n$. The subalgebra generated by $e_{n+1}, \dots, e_{n+8}$ is isomorphic to $C\ell_8 \cong \mathcal{M}_{16}(\mathbb{R})$. These two subalgebras commute with each other element-wise and generate all of $C\ell_{n+8}$. Finally, Proposition 13 in Section 2 gives us the result. $\square$

**Example 23.** To illuminate the above theorem, we shall work through the proof for the case $n = 1$.

First, take an orthonormal basis of $\mathbb{R}^9$, $\{e_1, e_2, \dots, e_9\}$. We set $e' = e_1 e_2 \cdots e_9$ and observe that

$$(e')^2 = (e_1 e_2 \cdots e_9)(e_1 e_2 \cdots e_9) = \ldots = (e_1 e_2 \cdots e_9)(e_9 e_8 \cdots e_1) = \ldots = -1.$$

Hence $e' = e_1 e_2 \cdots e_9 \in C\ell_9$ generates the subalgebra $\mathbb{C} \cong C\ell_1$.

Next, it is clear that $\{e_2, e_3, \dots, e_9\}$ is a basis for $\mathbb{R}^8$ and so the subalgebra generated by it is $C\ell_8$, by construction.

To show that these two algebras commute element-wise we are required to show that $e' e_i = e_i e'$ i.e. $(e_1 e_2 e_3 \cdots e_9) e_i = e_i (e_1 e_2 e_3 \cdots e_9)$ $(2 \le i \le 9)$, which follows from the properties of Proposition 18.

It remains to show that $\{e', e_2, e_3, \dots, e_9\}$ generates all of $C\ell_9$, this is achieved by noticing that $e_1$ can be recovered from $e'$ on post-multiplication by the element $-e_9 e_8 \cdots e_2$.

Putting this together we have

$$C\ell_9 \cong \mathcal{M}_{16}(C\ell_1) \cong \mathcal{M}_{16}(\mathbb{C}).$$

# 4. Further Theory and Applications of Clifford Algebras

We conclude this report by looking at the directions in which further study of Clifford algebras may lead, both in developing additional theory and exploring applications to other sciences.

4.1. **Division Algebras.** The alert reader may have noticed that although in our introduction to Clifford algebras we promised to generalise the complex and quaternion algebras into higher dimensions we have had to drop one of their nice properties: division.

Roughly speaking (and perhaps dangerously close to sounding patronising), a *division algebra* is an algebra where division is possible. We have defined our algebras to be associative. If we further assume that they are of finite dimension, then being a division algebra is equivalent to having no divisors of zero. Alternatively, every non-zero element has a multiplicative inverse, that is to say, every non-zero element is a unit.

There is nothing to stop us however, from defining this concept for a non-associative algebra. We say that a non-associative algebra is a *(non-associative) division algebra* if the operations of left and right multiplication by any non-zero element are invertible.

Remarkably, it turns out that there exist, up to isomorphism, only four real division algebras. Three of these are associative, they are the familiar real, complex and quaternion algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, respectively. The octonions are our fourth division algebra. They are eight dimensional over $\mathbb{R}$ and non-associative, although they do satisfy a weaker condition, called *alternative*.

Albert, in his book [Albt 1939], states that the study of linear algebras 'reached its zenith when the solution was found for the problem of determining all rational division algebras'. The standard construction of these four algebras is named after the mathematicians Cayley and Dickson. For a detailed account of this and the relation to Clifford's algebras we direct the reader to [Bz 2001], an excellent paper entitled 'The Octonions'.

4.2. **Spinors.** Let us look now at the concept of a *Spinor* (pronounced spin-or). The mathematics of these objects is central to the understanding of the quantum physics of basic particles – like protons, neutrons and electrons. Indeed, as Penrose states in [Pnrse 2004], 'ordinary solid matter could not exist without its consequences'.

Essentially, a spinor is an object that, on completion of a rotation through an angle of $2\pi$, turns into its negative. This may seem counterintuitive to our everyday experience, absurd even. In place of a formal definition, we will describe spinors by analogy.

Picture a book lying on a table in front of you. We want to keep track of the rotations of the book, so let's open it and place a belt in between the pages. Next, fix the buckle end of the belt (under another pile of books, say). This set-up is shown in Figure 7 (a). Now, rotating the book through $2\pi$ puts a twist in the belt, which cannot be undone without further rotation (as shown in (b)). But something curious happens if we rotate the book through another $2\pi$, the twist in the belt can be undone by looping the belt over the book (shown in (c)).

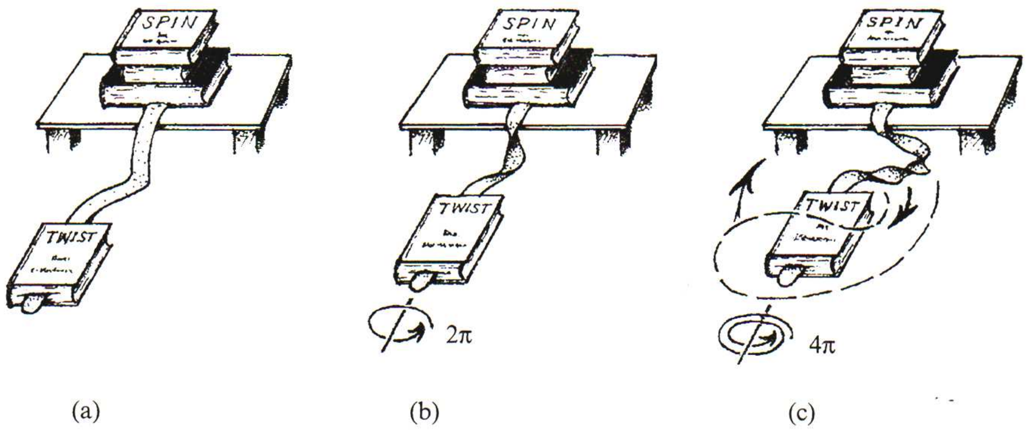


Figure 7. Spinorial Book

Thus, the belt keeps track of the parity of the number of $2\pi$ rotations. That is to say, if the book is rotated through an even number of $2\pi$ rotations the twist in the belt can be removed without further rotations of the book, whereas an odd number of $2\pi$ rotations leaves an inevitable twist in the belt. This holds true for any combination of rotations through any axes.

Spinors are connected with Clifford algebras in a very fundamental way, which we will illuminate at the end of the following section.

4.3. **Embedding $C\ell_n$ in $C\ell_{n+1}$.** As we noted in Section 3.2 (i), there is a canonical embedding of $C\ell_n$ in $C\ell_{n+1}$ given by the containment $\mathbb{R}^n \subset \mathbb{R}^{n+1}$.

A less obvious embedding can be found by considering

$$\mathbb{R}^n \longrightarrow C\ell_{n+1} \qquad x \mapsto xe_{n+1}.$$

It turns out that this can be extended to an algebra isomorphism

$$C\ell_n \longrightarrow C\ell^+_{n+1} \qquad e_i \mapsto e_i e_{n+1}$$

$(1 < i < n+1)$, where $C\ell_{\mathrm{n}}^{+}$ is the subalgebra of $C\ell_{\mathrm{n}}$ generated by a subset of the basis for $C\ell_{\mathrm{n}}$ consisting entirely of products of even length. For example, $C\ell_{2}^{+}$ has basis $\{1, e_1e_2\}$ and similarly, $C\ell_{3}^{+}$ has basis $\{1, e_1e_2, e_1e_3, e_2e_3\}$.

Returning to the concept of spinors, we shall make a couple of definitions to aid our explanation of the connections between these and Clifford algebras.

A closely related concept is that of *pinors*. Let $\mathrm{Pin}_n$ be the group of pinors sitting inside $C\ell_{\mathrm{n}}$, consisting of all the products of unit vectors in $\mathbb{R}^n$.

Consider now the following homomorphism between $\mathrm{Pin}_n$ and the orthogonal group, $\mathcal{O}_n$. For a unit vector $x \in \mathbb{R}^n$, we map both $\pm v$ to the element of $\mathcal{O}_n$ representing the reflection in the hyperplane perpendicular to $v$. Since every element of $\mathcal{O}_n$ is composed by reflections, it is easily seen that this homomorphism is surjective.

Moreover, $\mathrm{Pin}_n$ is a double cover for $\mathcal{O}_n$, that is to say every element of the orthogonal group is represented by two opposite pinors. Another way of expressing this is to say that the kernel of this mapping consists of just two elements, namely $\pm 1$. The concept of double covers should be familiar to the reader – you need look only as far as the hands on your wrist-watch! As each hand position corresponds to two positions of the sun.

Now, let $\mathrm{Spin}_n$ be the group consisting of all the products of an even number of unit vectors in $\mathbb{R}^n$. This group, a subgroup of $\mathrm{Pin}_n$, is the group of spinors as described in Section 4.2.

An element of $\mathcal{O}_n$ is also an element of $\mathcal{SO}_n$, the special orthogonal group, precisely when it is the product of an even number of reflections. Thus, just as $\mathrm{Pin}_n$ is a double cover of $\mathcal{O}_n$, $\mathrm{Spin}_n$ is a double cover of $\mathcal{SO}_n$.

As we know, every rotation in $\mathbb{R}^n$ can be represented by an element of $\mathcal{SO}_n$. The trouble is that elements of $\mathcal{SO}_n$ don't so much as represent rotations, but represent the final result of such a rotation – the sense of the rotation has been lost. This is where $\mathrm{Spin}_n$ enters and the veil shrouding its importance is lifted, as with one element of this group we can truly represent a rotation, its magnitude and sense.

Intriguingly, the epimorphism above from $\mathrm{Pin}_n$ to $\mathcal{O}_n$, also tells us that pinors in $n$ dimensions are spinors in $n+1$ dimensions!

4.4. **More Applications in Physics and Computing.** Next, a brief glance at some other applications of Clifford algebras. In 1928, the Dirac electron equation provided a turning point for physics. Dirac himself, unaware of Clifford and Hamilton's earlier work, was driven to reinvent parts of Clifford algebra in an attempt to understand the physical spin of the fundamental particles of nature.

To conclude the report let us take a brief look at some recent developments in computing. Here, Clifford algebras are presently being groomed for use in modelling geometries. These are in turn used in complex computer graphics. The current methods for modelling geometries, according to Dorst [Drst 2001], are fragmented at best.

> ...in every application a bit of linear algebra, a bit of differential geometry, a bit of vector calculus, each sensible used, but *ad hoc* in their connections ...

He goes on to say that this approach leads to unnatural splits in the program – rather than a division of tasks matching the nature of the problem. The solution he offers is a single 'Clifford' toolbox. Allowing calculations to be performed in a single framework, free from coordinates. Claiming also, that generalisation of programs to higher dimensions becomes intuitive under these constructions. Altogether, it seems as if he and his colleagues are trying to start a Clifford algebra revolution in computing!

# 5. Appendix

For the interested reader, here is the more general definition for a Clifford algebra that was promised in the preface.

**Definition 24.** Given an inner product space $(V, (\cdot|\cdot))$ and associated quadratic form $Q(x) = (x|x)$, there is a *Clifford algebra* $C\ell_{\mathrm{V}}(Q)$ which contains $V$ as a subspace and satisfies

$$x^2 = -Q(x)1$$

for all $x \in V$.

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