Smooth Exceptional del Pezzo Surfaces

Andrew Wilson

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Abstract

For a Fano variety V with at most Kawamata log terminal (klt) singularities and a finite group G acting bi-regularly on V, we say that V is *G-exceptional* (resp., *G-weakly-exceptional)* if the log pair (V, Δ) is klt (resp., log canonical) for all G-invariant effective $\mathbb{Q}$-divisors Δ numerically equivalent to the anti-canonical divisor of V. Such G-exceptional klt Fano varieties V are conjectured to lie in finitely many families by Shokurov ([Sho00, Pro01]). The only cases for which the conjecture is known to hold true are when the dimension of V is one, two, or V is isomorphic to n-dimensional projective space for some n. For the latter, it can be shown that G must be primitive — which implies, in particular, that there exist only finitely many such G (up to conjugation) by a theorem of Jordan ([Pro00]).

Smooth G-weakly-exceptional Fano varieties play an important role in non-rationality problems in birational geometry. From the work of Demailly (see [CS08, Appendix A]) it follows that Tian's α_G-invariant for such varieties is no smaller than one, and by a theorem of Tian such varieties admit G-invariant Kähler-Einstein metrics. Moreover, for a smooth G-exceptional Fano variety and given any G-invariant Kähler form in the first Chern class, the Kähler-Ricci iteration converges exponentially fast to the Kähler form associated to a Kähler-Einstein metric in the $C^\infty(V)$-topology. The term *exceptional* is inherited from singularity theory, to which this study enjoys strong links.

We classify two-dimensional smooth G-exceptional Fano varieties (del Pezzo surfaces) and provide a partial list of all G-exceptional and G-weakly-exceptional pairs (S, G), where S is a smooth del Pezzo surface and G is a finite group of automorphisms of S. Our classification confirms many conjectures on two-dimensional smooth exceptional Fano varieties.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Andrew Wilson*)

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Chapter 1

Introduction

The aim of this thesis is to decide which smooth del Pezzo surfaces with a regular action of some finite group G are G-exceptional and G-weakly-exceptional. One of our main applications also requires that the G-invariant Picard rank of these G-surfaces is one and so it is natural that our secondary aim is to identify all such smooth del Pezzo G-surfaces that are G-weakly-exceptional under this restriction. The classification of such pairs was recently completed by Dolgachev and Iskovskikh. We introduce this subject with a brief discussion of how to decide the exceptionality or non-exceptionality of a given G-surface.

Let (S,G) be a smooth del Pezzo G-surface where G is a finite group of automorphisms of S. Then how do we decide if (S,G) is exceptional or not? It is conjectured (Conjecture 16) that there is always a divisor, a 'wild tiger', realising the global G-invariant log canonical threshold (see Section 2.4). If this conjecture is true then the definitions of strongly-exceptional and G-exceptional coincide (Conjecture 29): In every case we calculated, this is the case. Consider the pluri-anti-canonical G-linear system $|-mK_S|^G$ where $m \in \mathbb{N}$ is the smallest such that $|-mK_S|^G$ is non-empty. To decide on the exceptionality of (S,G) we 'hunt for wild tigers':

Firstly we examine how the representation of G on $H^0\Big(S,\mathcal{O}_S(-mK_S)\Big) \cong \mathbb{C}^k$ splits into a direct sum of irreducible sub-representations, which yields candidates for our wild tigers. Secondly, we calculate their log canonical thresholds and hence the log canonical threshold of all divisors in $|-mK_S|^G$ ($\mathrm{lct}_m(S,G)$). Lastly, we prove that we do indeed have a wild tiger by showing that it realises the global log canonical threshold, that is that $\mathrm{lct}(S,G) = \mathrm{lct}_m(S,G)$. We direct the reader to some examples of this below.

Examples. Let S_d be a smooth del Pezzo of degree d (see Definition 19). Then

(i) (S_d, A) is non-exceptional for finite Abelian $A \leqslant \mathrm{Aut(S_d)}$ — Lemma 30 (p 17);

(ii) $(S_1, \mathbb{D}_8)$ is exceptional — Section 6.1.3.14 (p 77);

(iii) (S_4, G) is always weakly-exceptional for $\mathbb{Z}_2^4 \leqslant G \leqslant \mathrm{Aut(S_4)}$ — Lemma 170 (p 138);

(iv) $(S_5, \mathbb{Z}_5)$ is not weakly-exceptional — Lemma 184 (p 148);

(v) (S_6, G) is never exceptional for finite subgroups G of $\mathrm{Aut(S_6)}$ — Section 193 (p 160).

Chapter 2

Background

For the convenience of the reader and in the interests of self-containment, this chapter collects and presents some relevant material to provide background for the main results. More specifically, we introduce notions to measure singularities both locally and globally — discrepancies and the log canonical threshold, an algebraic counterpart of Tian's α-invariant. We also discuss briefly the equivariant minimal model program and the relation of this work to the study of conjugate subgroups of the group of birational transformations of the projective plane, $\mathrm{Cr}_2(\mathbb{C})$.

2.1 Singularities of Pairs and Discrepancy

For a normal variety[1] V, let $\Delta = \sum \delta_i \Delta_i$ be an effective $\mathbb{Q}$-Cartier divisor where the Δ_i are irreducible Weil divisors on V. Suppose that $K_V + \Delta$ is $\mathbb{Q}$-Cartier so that we may consider its numerical pull-back. Then for some variety U and a birational morphism $\varphi : U \longrightarrow V$ with exceptional divisors E_i we may write

$$K_U + \Delta_U \sim_{\mathbb{Q}} \varphi^*(K_V + \Delta) + \sum a(E_i; V, \Delta) E_i,$$

[1] All varieties are considered to be normal, projective and defined over a the field of complex numbers, $\mathbb{C}$ — unless explicitly stated otherwise.

where K_U, K_V are the canonical divisors on U, V, respectively; and Δ_U is the strict or proper transform of Δ on U. The number $a(E_k; V, \Delta)$ is called the *discrepancy* of the exceptional divisor E_k with respect to the log pair $\left(V, \Delta\right)$.

For any such birational morphism $\varphi : U \longrightarrow V$, we say that an irreducible divisor $E \subset U$ such that the image $\varphi(E)$ has co-dimension two or more is an *exceptional divisor over* V and the image $\varphi(E) \subset V$ is the *centre* of E on V. To get a global measure of the singularities of $\left(V, \Delta\right)$ we define:

Definition 1. The *discrepancy* of the log pair $\left(V, \Delta\right)$ is the number

$$\mathrm{discrep}\left(\mathrm{V}, \Delta\right) = \inf_{\mathrm{E}} \left\{ \mathrm{a}(\mathrm{E}; \mathrm{V}, \Delta) \,\middle|\, \mathrm{E \text{ is an exceptional divisor over } V} \right\}.$$

We impose restrictions on this global measure, defining several classes of pairs $\left(V, \Delta\right)$. For more details on these classes and their uses see [Kol97, KM98].

Definition 2. We say that $\left(V, \Delta\right)$, or $K_V + \Delta$ is

$$\left.\begin{array}{r} \text{terminal} \\ \text{canonical} \\ \text{Kawamata log terminal (klt)} \\ \text{purely log terminal (plt)} \\ \text{log canonical (lc)} \end{array}\right\} \text{ if } \mathrm{discrep}\left(\mathrm{V}, \Delta\right) \text{ is } \left\{\begin{array}{l} > 0, \\ \geqslant 0, \\ > -1 \text{ and } \lfloor \Delta \rfloor \leqslant 0, \\ > -1, \\ \geqslant -1. \end{array}\right.$$

For $\Delta = 0$, we say that V has terminal,canonical,log terminal or log canonical singularities if $\left(V, \Delta\right)$ has (the definitions for klt and plt coincide when the boundary $\Delta = 0$).

Remarks 3. Rather than working with all birational morphisms over V, to calculate the discrepancy of some pair $\left(V, \Delta\right)$ and determine which of the above classes is belongs, it suffices to understand the log resolution. The *log resolution* of the pair $\left(V, \Delta\right)$ is a birational morphism $\varphi : U \longrightarrow V$ such that U is smooth, the exceptional divisors of φ and all components of the strict transform of Δ are smooth and $\mathrm{Supp}(\Delta)$ is a simple normal crossings divisor. Hironaka's theorem on the resolution of singularities in characteristic zero gives us that log resolutions exist for all pairs $\left(V, \Delta\right)$.

Observe also that either discrep$(V, \Delta) = -\infty$, or discrep(V, Δ) lies between ± 1. That is, if the singularities of the pair (V, Δ) are worse than log canonical then the discrepancy can no longer provide a measure of their severity. For an example of this behaviour where the discrepancy of the pair is $-\infty$, and to see how we resolve measuring how bad their singularities are see Example 8 below.

Examples 4.

(i) Let $C \subset \mathbb{C}^2$ be an irreducible curve. Then the pair $(\mathbb{C}^2, C)$ is purely log terminal if, and only if, C is smooth and log canonical whenever C is nodal.

(ii) ([Kol97, Theorem 3.6]) For a germ of a normal surface singularity $(S \ni 0)$, S is

$$\left.\begin{array}{r} \text{terminal} \\ \text{canonical} \\ \text{Kawamata log terminal (klt)} \\ \text{log canonical (lc)} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \text{smooth;} \\ \mathbb{C}^2 \big/ \left(\text{ finite subgroup of } \mathbb{SL}_2(\mathbb{C})\right); \\ \mathbb{C}^2 \big/ \left(\text{ finite subgroup of } \mathbb{GL}_2(\mathbb{C})\right); \\ \text{simple elliptic, cusp, smooth,} \\ \quad \text{or a quotient of these by a finite group.} \end{array}\right.$$

We conclude this section with a very useful Lemma and Corollary, that we will use frequently in the proofs of Chapter 6. For further details on any of the above, we urge the reader to consult one of [Kol97, KM98, KSC04].

Lemma 5 (Convexity)**.** *Let V be a $\mathbb{Q}$-factorial variety, with Δ, Z effective $\mathbb{Q}$-divisors on V. Suppose that (V, Δ) and (V, Z) are log canonical. Then, for $\alpha \in \mathbb{Q}$ with $0 \leqslant \alpha \leqslant 1$,*

$$(V, \alpha\Delta + (1-\alpha)Z)$$

is log canonical.

Proof. Assume that $(V, \alpha\Delta + (1-\alpha)Z)$ is not log canonical, consider its log resolution $\overline{V} \longrightarrow V$ and compare discrepancies on $\overline{V}$. □

Corollary 6 (cf. [CS08, Remark 2.23])**.** *Let V be a $\mathbb{Q}$-factorial variety, with effective $\mathbb{Q}$-divisors $D = \sum_{i=1}^{r} d_i D_i$ and $Z = \sum_{i=1}^{r} z_i D_i$ where $d_i, z_i \in \mathbb{Q}_{\geqslant 0}$ and the D_i are prime Weil divisors for*

$i \in \{1,\dots,r\}$. Suppose that $D \sim_{\mathbb{Q}} Z$; the log pair $\big(V,Z\big)$ is log canonical at a point $P \in V$ and $\big(V,D\big)$ not log canonical at P. Write

$$\alpha = \min\left\{ \frac{d_i}{z_i} \,\Big|\, z_i \neq 0 \right\}.$$

Then α is well-defined as there are $z_i \neq 0$, $0 \leqslant \alpha < 1$ and since we may write $D = \alpha Z + (1-\alpha)\Delta$, where

$$\Delta = \sum_{i=1}^{r} \frac{d_i - \alpha z_i}{1-\alpha} D_i \sim_{\mathbb{Q}} D,$$

it follows from (the contra-positive of) Convexity (Lemma 5) that $\big(V,\Delta\big)$ is not log canonical and there exists at least one component of the divisor $\mathrm{Supp}(Z)$ *that is not contained in* $\mathrm{Supp}(\Delta)$.

Example 7. Our use of the Convexity Lemma can be seen in detail in the proof of Lemma 103 (after Figure 6.1, page 54).

2.2 Log Canonical Threshold

Lettuce begin with an example.

Example 8. Consider an irreducible curve C on $\mathbb{C}^2$ with a simple cuspidal point O (e.g. C is defined by $y^2 = x^3$), and let $\sigma_1, \sigma_2, \sigma_3$ be a series of blow-ups with exceptional divisors E_1, E_2, E_3, as in Figure 2.1. Writing $\sigma : X \longrightarrow \mathbb{C}^2$ for the composition of the maps σ_i, C_X for the strict transform of C on X, and by abusing notation E_i for the strict transforms of the E_i for $i = 1,2,3$; we see that σ is a log resolution and that

$$K_X + C_X = \sigma^*\big(K_{\mathbb{C}^2} + C\big) - E_1 - E_2 - 2E_3.$$

Thus $\mathrm{discrep}\big(\mathrm{E}_3;\mathbb{C}^2,\mathrm{C}\big) = -2$ and so $\big(\mathbb{C}^2, C\big)$ is not log canonical. Worse, if we blowup at a smooth point on E_3 not on E_1, E_2 or C_X and then continue to blowup at the intersection between E_3 and the new exceptional divisor F_k we see that

$$\mathrm{discrep}\big(\mathrm{F_k};\mathbb{C}^2,\mathrm{C}\big) \xrightarrow[\mathrm{k}\longrightarrow\infty]{} -\infty.$$

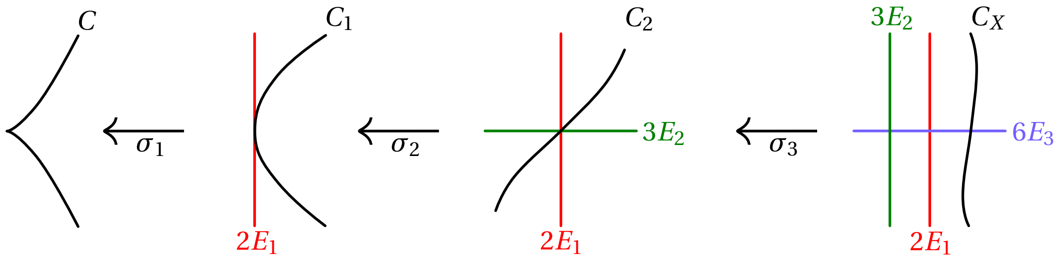


Figure 2.1: The log resolution of a cusp $C \subset \mathbb{C}^2$.

To rectify our failure to measure this singularity we observe that the pair $(\mathbb{C}^2, \lambda C)$ is log canonical for $\lambda = 0$ and not for $\lambda = 1$ — thus we may ask: How 'much' of C can we take for this singularity to jump back onto our scale in Definition 2? That is to say, what is the largest $\lambda \in \mathbb{Q}_{\geqslant 0}$ such that $(\mathbb{C}^2, \lambda C)$ is log canonical.

From the log resolution $\sigma : X \longrightarrow \mathbb{C}^2$, we have that

$$K_X + \lambda C_X = \sigma^*\left(K_{\mathbb{C}^2} + \lambda C\right) + (1-2\lambda)E_1 + (2-3\lambda)E_2 + (4-6\lambda)E_3.$$

As we want $\operatorname{discrep}(\mathbb{C}^2, \lambda \mathrm{C}) \geqslant -1$, it follows that $\lambda \leqslant \frac{5}{6}$. We call this number the log canonical threshold of $(\mathbb{C}^2, C)$ at the point O.

The log canonical threshold (lct) is an algebraic counterpart to the complex singularity exponent (cf. [Var82, Var83, Kol97, DK01]). Indeed, consider a polynomial $f \in \mathbb{C}[X_1, \ldots, X_n]$ with a singularity at a point P. To study the complexity of this singularity, a natural starting point is to look at the multiplicity of f at the point P which can be defined as

$$\operatorname{mult}_P(f) = \min\left\{ m \;\middle|\; \frac{\partial^m f}{\partial^{m_1} z_1 \partial^{m_2} z_2 \ldots \partial^{m_n} z_n}(P) \neq 0 \right\}.$$

Contrast this with the more subtle invariant, the *complex singularity exponent* of f at P, defined by integrations as follows

$$c_P(f) = \sup\left\{ c \;\middle|\; |f|^{-2c} \text{ is locally integrable near the point } P \in \mathbb{C}^n \right\}.$$

From [Kol97], it follows that

$$c_P(f) = \mathrm{lct}_P(\mathbb{C}^n, D),$$

where D is the divisor on $\mathbb{C}^n$ defined by the zeros of f and $\mathrm{lct}_P(\mathbb{C}^n, D)$ is the log canonical threshold at the point P of the pair $(\mathbb{C}^n, D)$, defined explicitly below.

Definition 9. Let V be a variety with at worst log canonical singularities, $Z \subseteq V$ a closed sub-variety and Δ an effective $\mathbb{Q}$-Cartier divisor on V. Then the *log canonical threshold* (*lct*) of the log pair (V, Δ) along Z is the number

$$\mathrm{lct}_Z(V, \Delta) = \sup\left\{\lambda \in \mathbb{Q} \,\middle|\, (V, \lambda\Delta) \text{ is log canonical along } Z\right\} \in \mathbb{Q} \cup \{+\infty\},$$

where $(V, \lambda\Delta)$ is log canonical *along* Z whenever $Z \not\subseteq \mathrm{LCS}(\mathrm{V}, \lambda\Delta)$ (see Definition 87). For our purposes, we shall only need this definition on the case where Z is a point.

We can also consider the log canonical threshold of Δ along the whole of V, $\mathrm{lct}_V(V, \Delta)$, which we write simply as $\mathrm{lct}(V, \Delta)$;

$$\begin{aligned}\mathrm{lct}(V, \Delta) &= \inf\left\{\mathrm{lct}_P(V, \Delta) \,\middle|\, P \in V\right\}\\ &= \sup\left\{\lambda \in \mathbb{Q} \,\middle|\, \text{ the log pair } (V, \lambda\Delta) \text{ is log canonical }\right\}.\end{aligned}$$

Example 10 ([CPS08, Example 1.1.3])**.** Let D be a cubic curve on the projective plane $\mathbb{P}^2$. Then

$$\mathrm{lct}(\mathbb{P}^2, D) = \begin{cases} 1 & \text{if } D \text{ is a smooth curve,}\\ 1 & \text{if } D \text{ is a curve with ordinary double points,}\\ \frac{5}{6} & \text{if } D \text{ is a curve with one cuspidal point,}\\ \frac{3}{4} & \text{if } D \text{ consists of a conic and a line that are tangent,}\\ \frac{2}{3} & \text{if } D \text{ consists of three lines intersecting at one point,}\\ \frac{1}{2} & \text{if } \mathrm{Supp}(D) \text{ consists of two lines,}\\ \frac{1}{3} & \text{if } \mathrm{Supp}(D) \text{ consists of one line.}\end{cases}$$

Remark 11. Let S be a smooth surface with some effective $\mathbb{Q}$-divisor Δ. Suppose that the pair (S, Δ) is not log canonical at some point P, but log canonical at all other points on S. Let

$\pi : Y \longrightarrow S$ be the blow-up of S at the point P, with exceptional divisor E. Write Δ_Y for the strict transform of Δ on Y. Then there exists a point $Q \in E$ such that the following inequality holds:

$$\mathrm{mult}_P\Delta + \mathrm{mult}_Q\Delta_Y > 2.$$

Indeed, if there are one-dimensional centres of log canonicity on Y then $\mathrm{mult}_P\Delta > 2$ and every point on E satisfies the inequality. Suppose then that $\mathrm{mult}_P\Delta \leqslant 2$. From the equivalence

$$\pi^*\left(K_S + \Delta\right) \equiv K_Y + \Delta_Y + (\mathrm{mult}_P\Delta - 1)E$$

we see that there must exist some point $Q \in E$ such that the pair $\left(Y, \Delta_Y + (\mathrm{mult}_P\Delta - 1)E\right)$ is not log canonical there. Observe then that as $\mathrm{mult}_P\Delta > 1$ ($\Delta_Y + (\mathrm{mult}_P\Delta - 1)E$ is effective), the above inequality holds.

2.3 G-Varieties

Definition 12. For a variety V of dimension n and a finite group G, we say

- $\left(V, G\right)$ (or simply V) is a G-*variety* if G acts bi-regularly on V — that is, $G \leqslant$ Aut(V);
- a morphism (resp., birational map) $\varphi : V \longrightarrow U$ is G-*equivariant* if the action of $\varphi \circ G \circ \varphi^{-1}$ on U is bi-regular, i.e. U is a $(\varphi \circ G \circ \varphi^{-1})$-variety. Thus, two subgroups of Aut(V) define isomorphic G-varieties if and only if they are conjugate in Aut(V);
- V is G-*rational* if there exists a G-equivariant birational map $\psi : V \dashrightarrow \mathbb{P}^n$.

2.4 G-Invariant Global Log Canonical Threshold

Let V be a Fano variety with at worst log terminal singularities, that is a variety where $-K_V$ is ample. Let also G be a finite[2] subgroup of the automorphism group of V and consider the Fano G-variety $\left(V, G\right)$.

[2]Though finite groups are enough for our porpoises here, we may define the G-invariant global lct when G is compact. However, in this case we must be careful to consider G-invariant *linear sub-systems* $\Delta \subset |-K_V|$ and not only G-invariant divisors (see [CS08, Definition 1.21]).

Definition 13 ([CS08, Section 1])**.** The *G-invariant global log canonical threshold* of (V,G) is defined to be the number

$$\mathrm{lct}(V,G) = \inf\Big\{\mathrm{lct}(V,\Delta)\,\Big|\,\Delta \text{ is an effective } G\text{-invariant } \mathbb{Q}\text{-divisor on } V \text{ such that } \Delta \sim_{\mathbb{Q}} -K_V\Big\}.$$

This is an algebraic counterpart to the α-*invariant* introduced by Tian (see [Tia87]). Moreover, for a smooth Fano variety V and a finite group G the equality

$$\alpha_G(V) = \mathrm{lct}(V,G)$$

holds ([CS08, Appendix A]).

Due to the rational connectedness of V ([Zha06]), we may re-formulate this definition as

$$\mathrm{lct}(V,G) = \sup\Big\{\lambda \in \mathbb{Q}\,\Big|\,\text{the log pair } (V,\lambda\Delta) \text{ is lc for all } G\text{-invariant } \mathbb{Q}\text{-divisors } 0 \leqslant \Delta \equiv -K_V\Big\}.$$

Remark 14. The above definition of the log canonical threshold is, in practise, difficult to work with. As we mentioned in the Introduction (Chapter 1) to calculate these thresholds we look in the pluri-anti-canonical linear systems for the 'worst' G-invariant divisors (i.e. those with the smallest log canonical threshold) and prove that they realise the global G-invariant log canonical threshold. It makes sense then to split the definition with an intermediate definition as follows.

Definition 15.

$$\mathrm{lct}_m(V,G) = \sup\Big\{\lambda \in \mathbb{Q}\,\Big|\,\text{ the log pair } (V,\frac{\lambda}{m}\Delta) \text{ is lc for all divisors } \Delta \in |-mK_V|^G\Big\}.$$

Then

$$\mathrm{lct}(V,G) = \inf\Big\{\mathrm{lct}_m(V,G)\,\Big|\,m \in \mathbb{N}\Big\} \geqslant 0.$$

Note that when $|-mK_V|$ contains no G-invariant divisors, $\mathrm{lct}_m(V,G)$ is defined to be $+\infty$.

Notation For a divisor H on V, we write $|H|^G$ for the set of all effective G-invariant divisors linearly equivalent to H, that is all the G-invariant members of $|H|$.

To date, no Fano varieties with non-rational global log canonical thresholds have been found. We expect this property to hold for all global log canonical thresholds. Furthermore, we expect that the global log canonical threshold is realised by a divisor in one of the pluri-anti-canonical linear systems (see [CPS08, Conjecture 1.1.11] and [Tia90a]). These divisors, numerically equivalent to the anti-canonical divisor, whose log canonical threshold realises the global log canonical threshold are called *wild tigers*. In this colourful language of Keel-MacKernan ([KM99]), we say that the calculation of global log canonical thresholds is, in part, the *hunt for wild tigers* (cf. [CP02]).

In this work we confirm, in every case we calculated, the following conjecture for the case where (X,G) is a smooth del Pezzo G-surface and G is finite.

Conjecture 16. *For a Fano variety V, let G be a finite subgroup of* Aut(V). *Then there exists an effective G-invariant $\mathbb{Q}$-divisor, $\Delta \sim_{\mathbb{Q}} -K_V$ such that*

$$\mathrm{lct}(V,G) = \mathrm{lct}(V,\Delta) \in \mathbb{Q}.$$

2.5 Minimal G-Surfaces and Conjugacy in the Cremona Groups

2.5.1 Rational varieties and the minimal model program

The minimal model program (MMP) is a method for choosing a 'good' representative for each birational class and for deciding which class any given variety lies. The simplest birational class is of course the rational varieties — varieties V which admit a birational map to the projective plane $V \dashrightarrow \mathbb{P}^n$. Recently in [BCHM10], this program was almost completed in all dimensions and has been the focus of much of the past few decades activity in algebraic geometry. Below we summarise *very* briefly the MMP and direct the reader to the well written paper [BCHM10], or the book [Mat02] for further details.

For a given smooth variety, V, the program has two possible outputs: Either a *minimal model*, a (possibly mildly singular) variety birational to V with numerically effective (nef)[3] canonical class; or a *Mori fibre space*, $V \longrightarrow S$, where S is projective, $\dim(V) > \dim(S)$ and

[3] An $\mathbb{R}$-Cartier divisor $\Delta \subset V$ is *numerically effective*, or *nef* if $\Delta \cdot C \geqslant 0$ for all irreducible curves $C \subset V$.

the anti-canonical divisor on a fibre is ample (so that the fibres are Fano).

For smooth surfaces, the theory is much simpler and was laid out in the early twentieth century by the Italian School. Unlike in higher dimensions, if we start with a smooth surface, our output is also smooth. For a smooth surface S, Castelnuovo's contractibility criterion tells us we may contract all the (-1)-curves on S. The result is either a (unique) minimal model, or ruled surface — that is, either a Mori fibre space from S to a point, i.e. $\mathbb{P}^2$; or a fibre space over a curve, a Hirzebruch surface $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)\right)$, with $n \neq 1$. Both these results can be collected under the label *minimal*: We call a surface S *minimal* if any birational map $\varphi : S \dashrightarrow T$ is in fact an isomorphism. For the rational surfaces the above can be summarised by the following classical proposition.

Proposition 17 ([Bea96, Theorem V.10])**.** *Let S be a minimal rational surface. Then S is isomorphic to $\mathbb{P}^2$, or to one of the Hirzebruch surfaces.*

Consider a group G acting biregularly on a smooth surface S, then we say that (S, G) is a *minimal G-surface* whenever any G-equivariant birational map $\varphi : S \dashrightarrow T$ is an isomorphism (cf. [Bla06, Definition 2.2.2],[DI10, Section 3.2]). Obviously this agrees with the usual definition of minimal when G is trivial. These minimal G-surfaces are the output of the G-equivariant minimal model program. In much the same way as without the group action, given a specific surface we contract G-orbits of disjoint exceptional curves on it obtaining a minimal G-surface (see [KM98, Example 2.18]). We have a similar answer as before by Dolgachev and Iskovskikh (cf. [Isk80, Theorem 1]).

Proposition 18 ([DI10, Theorem 3.8])**.** *Let S be a minimal rational G-surface*[4]*. Then either (S, G) admits a structure of a conic bundle with* $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S}) \cong \mathbb{Z}^2$, *or (S, G) is isomorphic to a del Pezzo G-surface with* $\mathrm{Pic}(\mathrm{S})^{\mathrm{G}} \cong \mathbb{Z}$.

Definition 19. A *del Pezzo* surface X is a Fano variety of dimension two. That is, a surface with ample anti-canonical divisor. The *degree* d_X of X is the number $K_X^2 \leqslant 9$, where K_X is the anti-canonical class. A smooth del Pezzo surface is rational and isomorphic to either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up in $9 - d_X$ points (see Section 5.1.1 for further details).

[4]Let S be a smooth minimal del Pezzo surface, then we assume in this thesis that $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S}) = \mathbb{Z}$, unless stated otherwise.

2.5.2 Conjugacy in the Cremona groups

The *Cremona group,* $\mathrm{Cr_n}(\mathbb{C})$, is the group of birational automorphisms of projective n-space, $\mathbb{P}^n_{\mathbb{C}}$. The Cremona group of the projective line is isomorphic to $\mathbb{PGL}_2(\mathbb{C})$. The Cremona group of the plane, $\mathrm{Cr}_2(\mathbb{C})$, is already large and complex. In higher dimensions, our understanding is even poorer. One step towards a holistic understanding of the structure of $\mathrm{Cr_n}(\mathbb{C})$ is the ability to describe the conjugacy classes (see Definition 98) of $\mathrm{Cr_n}(\mathbb{C})$. A modern approach to this, initiated by Iskovskikh and Manin (see e.g. [Man67, Isk80]), is to consider rational G-varieties and G-equivariant maps between them.

Theorem 20 ([DI10, Theorem 3.6])**.** *For a finite group $G \leqslant \mathrm{Cr_n}(\mathbb{C})$, there is a natural correspondence between G-equivariant birational isomorphism classes of rational G-varieties and conjugacy classes of subgroups of* $\mathrm{Cr_n}(\mathbb{C})$ *isomorphic to G.*

Sketch of Proof. Let V be a rational G-variety of dimension n.

- Since V is rational there is a birational map $\varphi : V \dashrightarrow \mathbb{P}^n$ that realises $G \leqslant \mathrm{Aut(V)}$ as a subgroup $\varphi G \varphi^{-1} \leqslant \mathrm{Cr_n}(\mathbb{C})$.
- For G-variety U and a G-equivariant birational map $U \dashrightarrow V$, U and V clearly define conjugate subgroups in $\mathrm{Cr_n}(\mathbb{C})$ (cf. Definition 12).
- For a finite subgroup $G \leqslant \mathrm{Cr_n}(\mathbb{C})$, there exists a smooth variety V and a birational map $\psi : V \dashrightarrow \mathbb{P}^n$ that 'resolves the indeterminacy of' ([dFE02]) or 'regularises' ([Che04]) G — that is, $\psi G \psi^{-1}$ acts regularly on V. □

For V a rational G-variety of dimension n with $G \leqslant \mathrm{Cr_n}(\mathbb{C})$ finite, two questions that naturally arise (cf. [Che09, Appendix B]) are:

Question 21. *Classify G (up to isomorphism) with some restriction (simple, Abelian, cyclic,...).*

Question 22. *For a given $G \leqslant \mathrm{Cr_n}(\mathbb{C})$ describe its conjugacy class.*

There are many existing answers to Questions 21 and 22 and as we will see in Section 4.1 the results of this thesis my be applied in certain situations to help decide on Question 22 (see Example 50).

Examples 23.

- Blanc in [Bla08] completes the classification of conjugacy classes of finite cyclic subgroups of $\mathrm{Cr}_2(\mathbb{C})$.
- It follows from the classification of finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ in [DI10] that if G is a finite simple non-Abelian subgroup of $\mathrm{Cr}_2(\mathbb{C})$ then $G \cong \mathbb{A}_5, \mathbb{A}_6$ or $\mathbb{PSL}_2(\mathbb{F}_7)$.
- Prokhorov shows in [Pro09] that if G is a finite simple non-Abelian subgroup of $\mathrm{Cr}_3(\mathbb{C})$, then $G \cong \mathbb{A}_5, \mathbb{A}_6, \mathbb{PSL}_2(\mathbb{F}_7), \mathbb{SL}_2(\mathbb{F}_8)$ or $\mathbb{SU}_2(\mathbb{F}_4)$.
- ([Che09, Theorem B.2]) The group $\mathrm{Cr}_2(\mathbb{C})$ contains 3,1,2 conjugacy classes of subgroups isomorphic to $\mathbb{A}_5, \mathbb{A}_6, \mathbb{PSL}_2(\mathbb{F}_7)$, respectively. The conjugacy classes can be represented by the G-surfaces $(\mathbb{P}^2, \mathbb{A}_5)$, $(S_5, \mathbb{A}_5)$, $(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{A}_5)$; $(\mathbb{P}^2, \mathbb{A}_6)$; $(\mathbb{P}^2, \mathbb{PSL}_2(\mathbb{F}_7))$ and $(S_2, \mathbb{PSL}_2(\mathbb{F}_7))$ respectively, where S_d is a smooth del Pezzo surface of degree d.

Observation 24. Let V be a rational n-dimensional variety and G a finite group acting regularly on V. Suppose that V is non-G-rational — that is, any birational map $\psi : V \dashrightarrow \mathbb{P}^n$ cannot be G-equivariant (Definition 12). Then clearly $\psi G \psi^{-1} \not\leqslant \mathrm{Aut}(\mathbb{P}^n)$. However since V is rational there is a birational map $\varphi : S \dashrightarrow \mathbb{P}^n$ and so $\varphi G \varphi^{-1} \leqslant \mathrm{Cr}_n(\mathbb{C})$ (and the choice of the map φ is independent of the conjugacy class of $\varphi G \varphi^{-1}$). So we conclude that $G \leqslant \mathrm{Cr}_n(\mathbb{C})$ is not conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^n)$.

Let us see how to apply this observation to see that $\mathrm{Cr}_2(\mathbb{C})$ contains at least two conjugacy classes of subgroups isomorphic to $\mathbb{PSL}_2(\mathbb{F}_7)$.

Example 25. Let S be the unique smooth del Pezzo surface of degree two with automorphism group $G = \mathbb{PSL}_2(7)$. Then by [DI10], $\mathrm{Pic}^G(S) = \mathbb{Z}$ and there is a birational map $\theta : S \dashrightarrow \mathbb{P}^2$. By Observation 24 it follows that $\theta G \theta^{-1}$ is a subgroup of $\mathrm{Cr}_2(\mathbb{C})$ but not $\mathrm{Aut}(\mathbb{P}^2)$. Furthermore, from [DI10], we also know that $G' = \mathbb{PSL}_2(7)$ acts regularly on $\mathbb{P}^2$ such that $(\mathbb{P}^2, G')$ is minimal. Hence G is not conjugate to G' in $\mathrm{Cr}_2(\mathbb{C})$.

In Section 4.1, we will see that together with a theorem of Pukhlikov and Cheltsov (Theorem 47) we can use this simple observation to answer similar questions of conjugacy in higher dimensional Cremona groups.

Chapter 3

Main Results

In this Chapter we introduce the notion of exceptionality and ask 'when is a smooth del Pezzo surface G-exceptional?' We also summarise what is currently known and present our results, the main of these being that except in degree six and the non-Kähler-Einstein del Pezzo surfaces (that is, the blowups of the projective plane in one or two points) there exist, in each degree, groups G such that smooth del Pezzo G-surfaces are G-exceptional.

3.1 Exceptionality

Let V be a Fano variety with at most klt singularities.

Definition 26. Then V is *G-exceptional* if there exists a finite group G acting biregularly on V such that the log pair (V, Δ) is klt for all G-invariant $\mathbb{Q}$-divisors

$$0 \leqslant \Delta \equiv -K_V.$$

For a given finite $G \leqslant \mathrm{Aut(V)}$, we say that the pair (V, G) is *exceptional* if it satisfies the above hypothesis.

Such G-exceptional klt Fano varieties are known to lie in finitely many families in dimensions one and two and conjectured to in higher dimensions ([Sho00, Pro01]). We also make two related definitions, those of weakly and strongly-exceptional Fano G-varieties.

Definition 27. For finite $G \leqslant \mathrm{Aut(V)}$, (V, G) is *weakly-exceptional* (respectively, *strongly-exceptional*) if the G-invariant log canonical threshold (Section 2.4), $\mathrm{lct}(V, G) \geqslant 1$ (respectively, $\mathrm{lct}(V, G) > 1$).

Remarks. Observe that:

- strongly exceptional $\Rightarrow$ exceptional $\Rightarrow$ weakly exceptional;
- (V, G) is weakly-exceptional but not strongly-exceptional $\Longleftrightarrow \mathrm{lct}(V, G) = 1$.

Due to Shokurov connectedness (Theorem 92), we have the following sufficient condition for weak-exceptionality on certain del Pezzo G-surfaces.

Lemma 28. *Let S be a smooth del Pezzo G-surface with G finite and suppose that*

(i) there are no G-fixed points on S and;

(ii) $\mathrm{Pic^G(S)} = \mathbb{Z}$ and is generated by the anti-canonical class.

Then S is G-weakly-exceptional.

Proof. Suppose that *(i)* and *(ii)* hold but that $\mathrm{lct}(S, G) < 1$. Let $\lambda \in \mathbb{Q}$ such that $\mathrm{lct}(S, G) < \lambda < 1$. Then there exists an effective G-invariant $\mathbb{Q}$-divisor $D = \sum_{i=0}^{r} d_i D_i \equiv -K_S$ where the D_i are prime Weil divisors and $(S, \lambda D)$ is not log canonical. By Shokurov Connectedness (Theorem 92), $\mathrm{LCS(S, \lambda D)}$ is connected. If $\mathrm{LCS(S, \lambda D)}$ is zero-dimensional, then it is a point — but this violates *(i)*. Thus it is one-dimensional and so there is a d_k such that $\lambda d_k > 1$. Writing $D = d_k(\Delta_1 + \cdots + \Delta_k) + \Omega$, where $\Delta_1 + \cdots + \Delta_k$ is a G-orbit of $\Delta_k = D_k$ and Ω is a one-cycle on S whose support doesn't contain the G-orbit $\Delta_1 + \cdots + \Delta_k$. Clearly $\Delta_1 + \cdots + \Delta_k \in |-\mu K_S|^G$ for some $\mu \in \mathbb{Z}_{>0}$ by *(ii)*. However, intersecting λD with $-K_S$ leads to a contradiction:

$$\lambda K_S^2 = \lambda D \cdot (-K_X) \geqslant \mu \lambda d_k K_S^2 > \mu K_S^2.$$

That is, $1 > \lambda > \mu$ which implies that $\mu \leqslant 0$. $\square$

The truth of Conjecture 16 would imply the following.

Conjecture 29. *(V,G) exceptional $\Longleftrightarrow$ (V,G) strongly-exceptional.*

Suppose that V is a $\mathbb{Q}$-factorial variety with finite automorphism group, G then we already know a large class of G-varieties that are never G-exceptional.

Lemma 30. *Suppose that G is a finite Abelian group and that $|-K_V| \neq \emptyset$, then there exist G-invariant curves in $|-K_V|$.*

Proof. The group G acts naturally on the space $H^0(V,\mathcal{O}_V(-K_V))$ which is isomorphic to $\mathbb{C}^n$, for some n. As G is Abelian, the representation of G on $\mathbb{C}^n$ splits as a direct sum of 1-dim sub-representations (see for example, [JL01, Thm 9.8]). Each of these corresponds to a 1-dim irreducible subspace of $\mathbb{C}^n$ — that is, curves on V belonging to $|-K_V|^G$. □

Remark 31. If the action of G on $\mathbb{C}^n$ is such that Eigenvalues of k sub-representations agree, then we have a G-invariant dimension k sub-linear system of $|-K_V|$.

Corollary 32. *The G-variety V is never G-exceptional whenever G is Abelian.*

3.2 Questions

We have two main applications for our smooth exceptional del Pezzo G-surfaces of degree d, (S_d,G). One birational in flavour (Section 4.1) and the other regarding Kähler geometry (Section 4.2). For S_d G-weakly-exceptional, Theorem 52 tells us that these G-surfaces admit a Kähler-Einstein metric and moreover by Theorem 56 the Kähler-Ricci flow converges. If S_d is G-strongly-exceptional then also the Kähler-Ricci iteration converges (Theorem 58). We may also answer certain questions of conjugacy in higher rank Cremona groups with the aid of Theorem 47 and Observation 24 whenever (S_d,G) is a minimal G-weakly-exceptional G-birationally super-rigid G-surface. For details see Chapter 4. Bearing these various applications in mind we ask:

Question A. *For a fixed degree d, when does there exist a finite group $G \leqslant \mathrm{Aut}(S_d)$ such that the pair (S_d,G) is G-exceptional or G-weakly-exceptional?*

Question B. *For a fixed degree d, what are all the possible finite groups $G \leqslant \mathrm{Aut}(S_d)$ such that the pair (S_d,G) is G-exceptional or G-weakly-exceptional?*

Question B is too large in scope for us to handle here, so we narrow the focus by restricting ourselves to considering only minimal smooth del Pezzo G-surfaces[1]. This is a natural restriction as the application of Theorem 47 requires $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S_d}) = \mathbb{Z}$ which is satisfied by minimal pairs $\left(S_d, G\right)$ that are not conic bundles (cf. Theorem 18).

Question C. *For a fixed degree d, which of the possible finite groups $G = \mathrm{Aut}(\mathrm{S_d})$ such that $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S_d}) = \mathbb{Z}$ yield a G-exceptional or G-weakly-exceptional pair $\left(S_d, G\right)$?*

Informally, we'll look at the Dolgachev-Iskovskikh ([DI10]) classification of finite subgroups of the Cremona group $\mathrm{Cr}_2(\mathbb{C})$. There they realise each of the subgroups as an G-action on a smooth del Pezzo surface S_d with $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S_d}) = \mathbb{Z}$ (fibrations) or $\mathbb{Z} \oplus \mathbb{Z}$ (conic bundles). Thus we have a list of all pairs (S_d, G) for S_d a smooth del Pezzo G-surface, $G = \mathrm{Aut}(\mathrm{S_d})$ and $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S_d}) = \mathbb{Z}$ for which we can calculate global G-log canonical thresholds.

3.3 Existing Answers

There exist previous partial answers to our Question A in the literature. In [Che08], we find the calculation of global log canonical thresholds for all smooth del Pezzo surfaces without the action of a group.

Theorem 33 ([Che08])**.** *Let S be a smooth del Pezzo surface. Then*

$$\mathrm{lct}\big(S\big) = \begin{cases} 1 \text{ when } K_S^2 = 1 \text{ and } |-K_S| \text{ has no cuspidal curves,} \\ 5/6 \text{ when } K_S^2 = 1 \text{ and } |-K_S| \text{ has a cuspidal curve,} \\ 5/6 \text{ when } K_S^2 = 2 \text{ and } |-K_S| \text{ has no tacnodal curves,} \\ 3/4 \text{ when } K_S^2 = 2 \text{ and } |-K_S| \text{ has a tacnodal curve,} \\ 3/4 \text{ when } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points,} \\ 2/3 \text{ when } K_S^2 = 4 \text{ or } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point,} \\ 1/2 \text{ when } S \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_S^2 \in \{5,6\}, \\ 1/3 \text{ in the remaining cases.} \end{cases}$$

[1]Let S be a smooth minimal del Pezzo surface, then we assume in this thesis that $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S}) = \mathbb{Z}$, unless stated otherwise.

Moreover in the same paper, Cheltsov determines the G-invariant global log canonical threshold of some smooth del Pezzo G-surfaces, partially answering Question A. To be precise, we summarise what is calculated in [Che08] in the following — details to be found in Section 5 of [Che08] or the relevant section of Chapter 6.

Proposition 34. *Let S_d be a smooth del Pezzo surface of degree d with prescribed automorphism group G. Then*

- $\mathrm{lct}(S_3, G) = \begin{cases} 2 \text{ when } G = \mathbb{A}_5 \text{ or } \mathbb{S}_5, \\ 4 \text{ when } G = \mathbb{Z}_3^3 \rtimes \mathbb{S}_4; \end{cases}$
- $\mathrm{lct}(S_5, G) = \begin{cases} \frac{4}{5} \text{ when } G = \mathbb{Z}_5, \\ 2 \text{ when } G = \mathbb{A}_5 \text{ or } \mathbb{S}_5; \end{cases}$
- $\mathrm{lct}(S_6, G) \leqslant 1$.
- $\mathrm{lct}(\mathbb{P}^2, G) = \begin{cases} \frac{2}{3} \text{ when } G = \mathbb{A}_5, \\ \frac{4}{3} \text{ when } G = \mathbb{PGL}_2(\mathbb{F}_7), \\ 2 \text{ when } G = \mathbb{A}_6; \end{cases}$

In fact, for $(\mathbb{P}^2, G)$ the work of Markushevich-Prokhorov ([MP99b, MP99a, Pro00]) in combination with that of Cheltsov-Shramov ([CS09]) completely answers Questions A and B in the exceptional case. Using the Miller-Blichfeldt-Dickson classification of finite subgroups of $\mathbb{GL}_3(\mathbb{C})$ ([BDM16]), Markushevich-Prokhorov give a classification of the groups $G \leqslant \mathbb{PGL}_3(\mathbb{C})$ for which the pair $(\mathbb{P}^2, G)$ is exceptional. Recently, Sakovics in [Sak10] extended these results classifying also those weakly-exceptional $(\mathbb{P}^2, G)$, $G \leqslant \mathbb{PGL}_3(\mathbb{C})$. We present a summary of these works in Sections 4.3 and 6.9.

3.4 Main Results

In this thesis, we answer completely Question A and partially Question C of Section 3.2. More precisely, we answer Question C completely for smooth del Pezzo G-surfaces of degrees $1, \ldots, 5$ for in both the G-exceptional and G-weakly-exceptional cases. In degree six, we give a necessary condition for G-weak-exceptionality but do not provide a list of such groups. For the non-Kähler-Einstein del Pezzo G-surfaces we show that their global G-lct at most two-thirds, and hence they cannot be G-weakly-exceptional. For $\mathbb{P}^1 \times \mathbb{P}^1$, we obtain a partial answer and give a scheme we hope will furnish a list of groups to answer Question C in full. Finally, for the projective plane, Question C is completely answered in the G-exceptional case by Markushevich-Prokhorov and Cheltsov-Shramov as we mentioned above; and in the G-weakly-exceptional recently by Sakovics. We present these results below.

Theorem 35. *Let S_d be a general[2] smooth minimal del Pezzo G-surface of degree $d = K_{S_d}^2 \leqslant 5$ with prescribed automorphism group G. Then*

$$S_d \text{ is } G\text{-exceptional} \iff \begin{cases} d=1,\ G = \mathbb{D}_8,\ \mathbb{Z}_2{\bullet}\mathbb{D}_4,\ \mathbb{D}_{12},\ \mathbb{D}_{16},\ \mathbb{Z}_2{\bullet}\mathbb{D}_{12},\ \mathbb{Z}_2{\bullet}\mathbb{A}_4,\ \mathbb{Z}_3\times\mathbb{D}_8, \\ \qquad\qquad \mathbb{Z}_2\times\mathbb{Z}_3{\bullet}\mathbb{D}_6,\ \mathbb{Z}_6{\bullet}\mathbb{D}_{12},\ \mathbb{Z}_3\times\mathbb{Z}_2{\bullet}\mathbb{S}_4; \\ d=2,\ G=\mathbb{S}_4\times\mathbb{Z}_2,\ \left(\mathbb{Z}_4^2\rtimes\mathbb{S}_3\right)\times\mathbb{Z}_2,\ \mathbb{PSL}_2(\mathbb{F}_7)\times\mathbb{Z}_2; \\ d=3,\ G=\mathbb{S}_5,\ \mathbb{Z}_3^3\rtimes\mathbb{S}_4; \\ d=4,\ G=\mathbb{Z}_2^4\rtimes\mathbb{S}_3,\ \mathbb{Z}_2^4\rtimes\mathbb{D}_{10}; \\ d=5,\ G=\mathbb{S}_5,\ \mathbb{A}_5. \end{cases}$$

$$\begin{array}{r} S_d \text{ is } G\text{-weakly-exceptional} \\ \text{— but not } G\text{-exceptional} \end{array} \iff \begin{cases} d=1,\ G=\mathbb{Z}_2,\ \mathbb{Z}_2\times\mathbb{Z}_2,\ \mathbb{Z}_4,\ \mathbb{Z}_6,\ \mathbb{Z}_2\times\mathbb{Z}_6,\ \mathbb{Z}_2\times\mathbb{Z}_{12}; \\ d=2,\ G=\mathbb{Z}_2,\ \mathbb{Z}_2\times\mathbb{Z}_2,\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,\ \mathbb{S}_3\times\mathbb{Z}_2,\ \mathbb{D}_8\times\mathbb{Z}_2, \\ \qquad\qquad \left(\mathbb{Z}_2\rtimes\mathbb{D}_8\right)\times\mathbb{Z}_2,\ \mathbb{Z}_4{\bullet}\mathbb{A}_4\times\mathbb{Z}_2; \\ d=3,\ G=\mathbb{S}_3,\ \mathbb{S}_3\times\mathbb{Z}_2,\ \mathbb{S}_4,\ \mathbb{Z}_3\left(\mathbb{Z}_2\rtimes\mathbb{Z}_3^2\right),\ \mathbb{Z}_3\left(\mathbb{Z}_4\rtimes\mathbb{Z}_3^2\right); \\ d=4,\ G=\mathbb{Z}_2^4,\ \mathbb{Z}_2^4\rtimes\mathbb{Z}_2,\ \mathbb{Z}_2^4\rtimes\mathbb{Z}_4; \\ d=5,\ G=\mathbb{Z}_5\rtimes\mathbb{Z}_4. \end{cases}$$

Proof. See Chapter 6; Theorems 104, 133, 151, 173, 183. □

Corollary 36. *For all $d \leqslant 5$, there exists S_d — a smooth minimal del Pezzo surface of degree d that is $\left(\mathrm{Aut}(\mathrm{S_d})\right)$-exceptional.*

Remark 37. For $d \leqslant 5$, $\mathrm{Aut}(\mathrm{S_d})$ is finite by Lemma 78.

We also summarise what we know in the case where the degree of our smooth del Pezzo is greater than five; further details can be found in the relevant section of Chapter 6.

Lemma 38. *Let S_6 be a smooth del Pezzo G-surface of degree six and $G \leqslant \mathrm{Aut}(\mathrm{S_6})$, then*

$$\mathrm{lct}\left(S_6, G\right) \leqslant 1.$$

[2]The required generality is made explicit in restrictions on the parameters of the defining equations of S_d — see the relevant section of Chapter 6, or Tables 7.2–7.6.

Proof. There are exactly six (-1)-curves on S_6 (see Table 5.2) and thus the divisor formed by their sum is a G-invariant member of $|-K_{S_d}|$ for any G acting biregularly on S_6 (see Section 6.6). □

Corollary 39 (to the proof of Lemma 38)**.** *There are no G-exceptional smooth del Pezzo G-surfaces of degree six.*

We have the following criterion for deciding on the G-weak-exceptionality of a smooth del Pezzo G-surface of degree six — but not a classification.

Theorem 40 (Theorem 194)**.** *For a smooth del Pezzo G-surface S_6 of degree six such that* $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S}_6) = \mathbb{Z}$,

$$\mathrm{lct}\big(S_6, G\big) = 1$$

if, and only if, $\big(S_6, G\big)$ has no G-fixed points.

Non-Kähler-Einstein del Pezzo G-surfaces are of course non-G-exceptional. Indeed, let S be the blowup of $\mathbb{P}^2$ in one or two points and observe that by Theorem 53, S does not admit a Kähler-Einstein metric and hence by the contra-positive of Theorem 52, $\mathrm{lct}\big(S, G\big) \leqslant \frac{2}{3}$. Therefore the G-surface S is not G-weakly-exceptional for any G. Furthermore, by Theorem 33, $\mathrm{lct}(S) = \frac{1}{3}$ and thus

$$\frac{1}{3} \leqslant \mathrm{lct}\big(S, G\big) \leqslant \frac{2}{3}.$$

Theorem 41 (cf. Section 6.7)**.** *Let S be the blowup of $\mathbb{P}^2$ in two points, then* $\mathrm{lct}(S, G) = \frac{1}{3}$ *for any G acting bi-regularly on S.*

Proof. If $S = S_7$ is the blowup of two points $Q_1, Q_2 \in \mathbb{P}^2$ with exceptional curves F_1, F_2 over Q_1, Q_2 respectively, then there is a G-invariant divisor in $|-K_{S_7}|$. Indeed, we may write $-K_{S_7} = 3M + 2F_1 + 2F_2$ where M is the strict transform under the blowup map of the line between the points Q_1, Q_2 and it follows that $\mathrm{lct}\big(S_7, G\big) \leqslant \frac{1}{3}$. □

For $\mathbb{P}^1 \times \mathbb{P}^1$, we present below a partial answer to our Questions of Section 3.2. We hope to answer fully our questions with a scheme we present in Section 6.8.1.

Corollary 42 (Theorem 201)**.** *Let A be a finite subgroup of $\mathbb{PGL}_2(\mathbb{C})$ such that*

$$\operatorname{lct}\big(\mathbb{P}^1 \times \mathbb{P}^1, A \times A\big) > 1.$$

Then $A \cong \mathbb{A}_5, \mathbb{A}_4$ or $\mathbb{D}_{2n}$ for some $n \in \mathbb{N}$.

As we mentioned in the previous section (Section 3.3) for the projective plane the G-exceptional version of Question B is completely answered by Markushevich-Prokhorov and Cheltsov-Shramov (see Sections 4.3 and 6.9). For the G-weakly-exceptional version of Question B we have laid out in Section 6.9.1 what is required for a complete answer (in fact, this was recently done by Sakovics in [Sak10]).

By Proposition 64 ([MP99b, Corollary 2.3]) for a finite group $G \leqslant \mathbb{GL}_3(\mathbb{C})$ without reflections (which we may always assume — see Section 4.3) if $\mathbb{P}^2$ is $\big(\pi(G)\big)$-exceptional then G is primitive, where $\pi : \mathbb{GL}_3(\mathbb{C}) \longrightarrow \mathbb{PGL}_3(\mathbb{C})$ is the natural map.

Proposition 43 (Proposition 205)**.** *Let G be a primitive subgroup of $\mathbb{GL}_3(\mathbb{C})$. Then*

$$\operatorname{lct}\big(\mathbb{P}^2, \pi(G)\big) \begin{cases} \leqslant \frac{2}{3} & \text{if } G \text{ is of type } H \text{ — i.e. } \pi(G) \cong \mathbb{A}_5, \\ \leqslant 1 & \text{if } G \text{ is of type } E \ (|\pi(G)| = 36), \\ > 1 & \text{otherwise (see Proposition 204 for a list of possibilities).} \end{cases}$$

For full details on the groups of type E and H see [BDM16, Section 115]. We summarise in the proof of Proposition 205 what is described in detail there.

Chapter 4

Applications of Main Results

Here we take the opportunity to present various applications of our results and some correspondences to other areas of mathematics. In particular, we see how to apply our classification of G-weakly-exceptional smooth del Pezzo G-surfaces with Picard rank one to problems of conjugacy in higher rank Cremona groups. It is known, by a result of Tian, that on all G-strongly-exceptional smooth G-Fano varieties there exists a G-invariant Kähler-Einstein metric. Moreover, recently Rubinstein showed that given any Kähler form in the first Chern class, the Kähler-Ricci iteration converges exponentially fast to the Kähler form associated to a Kähler-Einstein metric in the $C^\infty(V)$-topology. Lastly, we examine the correspondence with quotient singularities, where we see that our Questions A and B of Section 3.2 are answered for $\mathbb{P}^2$.

4.1 Conjugacy in Higher Rank Cremona Groups

In Section 2.5.2 we discussed conjugacy in the Cremona groups. In particular we observed (Observation 24) that if we find rational varieties of dimension n that are non-G-rational (Definition 12) for a finite group G then we can conclude that G is not conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^n)$, whilst of course belonging to a conjugacy class of $\mathrm{Cr}_n(\mathbb{C})$. Below, using Theorem 47 and our classification of smooth minimal G-weakly-exceptional del Pezzo surfaces, we see how to apply this observation to subgroups of $\mathrm{Cr}_{2k}(\mathbb{C})$ for $k \in \mathbb{N}$. First, let us make some definitions needed to that end.

Definition 44. A Fano G-variety V is $G\mathbb{Q}$-*Fano* if

(i) V has at worst terminal singularities,

(ii) all G-invariant Weil divisor on V are $\mathbb{Q}$-Cartier ($G\mathbb{Q}$-factorial) and,

(iii) $\mathrm{Pic}^{\mathrm{G}}(\mathrm{V}) = \mathbb{Z}$.

Of course all smooth minimal del Pezzo G-surfaces satisfy the above requirements and so are $G\mathbb{Q}$-Fano.

Definition 45. A $G\mathbb{Q}$-Fano variety V is G-*birationally-rigid* (G-BR) if

(i) there are no other $G\mathbb{Q}$-Fano varieties G-equivariantly birational to V and,

(ii) there are no G-equivariant birational map from V to a variety U such that there is a (non-birational) G-equivariant epimorphism $\xi : U \longrightarrow Z$ where $\dim(U) > \dim(Z) \neq 0$ and whose general fibre is an irreducible rationally connected variety.

(In particular, V G-BR $\Rightarrow$ V non-G-rational).

If in addition to the above we also have that $\mathrm{Bir}^{\mathrm{G}}(\mathrm{V}) = \mathrm{Aut}^{\mathrm{G}}(\mathrm{V})$ then we say that V is G-*birationally-super-rigid* (G-BSR).

Equivalently via the Nöether-Fano inequality (see [CS08, Theorem 1.26]), a $G\mathbb{Q}$-Fano variety V is G-BSR if for all G-invariant linear systems $\mathcal{M}$ on V that have no fixed components, the singularities of the log pair $(V, \lambda\mathcal{M})$ are canonical, where $\lambda \in \mathbb{Q}$ and $K_V + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$.

Examples 46.

- A smooth del Pezzo $\mathbb{A}_5$-surface of degree 5 is $\mathbb{A}_5$-BSR by Lemma 48.
- $\mathbb{P}^2$ is $\mathbb{A}_5$-BR but not $\mathbb{A}_5$-BSR as $\mathbb{A}_5 \cong \mathrm{Aut}^{\mathbb{A}_5}(\mathbb{P}^2) \subsetneq \mathrm{Bir}^{\mathbb{A}_5}(\mathbb{P}^2) \cong \mathbb{S}_5$ (see [Che09, Theorem B.12]).
- Let G be trivial and X quartic three-fold with nodal singularities (ordinary double points) such that the rank of the class group of X is one (which happens if $|\mathrm{Sing}(X)| \leqslant 8$). Then X is BR and X is BSR if X is smooth ([IM71, Puk88, Cor00, Mel04]).

Further details can be found in, for example, Cheltsov's survey 'Birationally rigid Fano varieties' ([Che05]); [Che09, Appendix A]; or [Cor00].

The following result is an equivariant version of [Puk05, Theorem 1] (cf. [CS08, Theorem 1.28], [Che09, Theorem A.29]).

Theorem 47. *For $1 \leqslant i \leqslant r$ with $r \geqslant 2$, let $V_1, \dots, V_r$ be G_i-weakly-exceptional G_i-BSR $G_i\mathbb{Q}$-Fano varieties. Write $\Gamma = G_1 \times \cdots \times G_r$ and $\Omega = V_1 \times \ldots \times V_r$.*

Then the Γ-variety Ω is non-Γ-rational,

$$\mathrm{Bir}^{\Gamma}(\Omega) = \mathrm{Aut}^{\Gamma}(\Omega)$$

and for every Γ-equivariant rational dominant map

$$\rho\colon \Omega \dashrightarrow \Psi,$$

whose general fibre is an irreducible rationally connected variety, there is a commutative diagram

$$\begin{tikzcd} \Omega \arrow[d, "\pi"'] \arrow[dr, dashed, "\rho"] & \\ V_{i_1} \times \ldots \times V_{i_k} \arrow[r, dashed, "\psi"'] & \Psi \end{tikzcd}$$

for some $\{i_1, \dots, i_k\} \subseteq \{1, \dots, r\}$, where ψ is a $\left(G_{i_1} \times \ldots \times G_{i_k}\right)$-equivariant birational map, and π a natural projection.

To apply the preceding theorem with Observation 24 and our classification of smooth minimal G-weakly-exceptional del Pezzo G-surfaces (Theorem 35) we must determine which of those G-weakly-exceptional minimal del Pezzo G-surfaces are G-BSR.

Lemma 48 (c.f. [Che05, Theorem 1.5.1], [DI10, Corollary 7.11], [Che09, Lemma A.19])**.** *Let S be a smooth del Pezzo G-surface of degree $d = K_S^2$ such that $\mathrm{Pic}^{\mathrm{G}}(\mathrm{S}) = \mathbb{Z}$, if for every G-orbit Σ on S*

$$|\Sigma| \geqslant d,$$

then S is G-BSR.

Proof. Suppose that S is not G-BSR. Then there is a G-invariant linear system $\mathcal{M}$ with no fixed curves but $(S, \lambda\mathcal{M})$ is not canonical at some point $P \in S$, where $\lambda \in \mathbb{Q}$ and $K_S + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$. Let Σ be the G-orbit of P, then for all $Q \in \Sigma$,

$$\operatorname{mult}_Q(\mathcal{M}) > \frac{1}{\lambda}$$

and hence

$$\frac{1}{\lambda}K_S^2 = M_1 \cdot M_2 \geqslant \sum_{Q \in \Sigma} \operatorname{mult}_Q^2(\mathcal{M}) > \frac{1}{\lambda}|\Sigma| \geqslant \frac{1}{\lambda}K_S^2,$$

for M_1, M_2 sufficiently general curves in $\mathcal{M}$. □

Corollary 49.

- *All minimal degree one smooth del Pezzo G-surfaces are G-BSR.*

- *All minimal degree two smooth del Pezzo G-surfaces are G-BSR, whenever G acts without fixed points (these are also G-weakly-exceptional by Lemma 28).*

- *Let S be a del Pezzo surface of degree five. Then $(S, \mathbb{A}_5)$ is minimal by [DI10] and S is $\mathbb{A}_5$-BSR since for all $\mathbb{A}_5$-orbits Σ on S, $|\Sigma| \geqslant 6$. Indeed, if $|\Sigma| < 6$ then by the Orbit-Stabiliser theorem $|\Sigma| = 5$ as $\mathbb{A}_5$ acts without fixed points and has no subgroups of orders $30, 20$ or 15. Let $H \leqslant \mathbb{A}_5$ be the stabiliser of some point $P \in \Sigma$, which acts faithfully on the tangent space of the point P. It follows that $|H| = 12$ and so $H \cong \mathbb{A}_4$ — but $\mathbb{A}_4$ does not have any faithful two-dimensional representations.*

- *$(\mathbb{P}^2, \mathbb{A}_6)$ is minimal by [DI10] and $\mathbb{P}^2$ is $\mathbb{A}_6$-BSR since $|\Sigma| \geqslant 12$ for all G-orbits Σ on $\mathbb{P}^2$ ([Spr77, YY93]).*

It would be interesting to complete the list of those G-weakly-exceptional minimal del Pezzo G-surfaces that are G-BSR.

Example 50 ([Che09, Lemma A.31])**.** The simple group $\mathbb{A}_6$ is a group of automorphisms of the sextic

$$10x^3y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2y^2z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \operatorname{Proj}\Big(\mathbb{C}[x, y, z]\Big)$$

which induces an embedding $\mathbb{A}_6 \leqslant \mathrm{Aut}(\mathbb{P}^2)$ such that $\mathrm{lct}\big(\mathbb{P}^2, \mathbb{A}_6\big) = 2$ by Proposition 34, and $\mathbb{A}_6 \times \mathbb{A}_6$ acts naturally on $\mathbb{P}^2 \times \mathbb{P}^2$. There is an induced embedding $\mathbb{A}_6 \times \mathbb{A}_6 \cong \Omega \leqslant \mathrm{Bir}(\mathbb{P}^4) \cong \mathrm{Cr}_4(\mathbb{C})$ such that Ω is not conjugated to a subgroup of $\mathrm{Aut}(\mathbb{P}^4)$ by Example 48 and Theorem 47.

4.2 Existence of Kähler-Einstein metrics and convergence of the Kähler-Ricci iteration

On Fano manifolds, global log canonical thresholds play an important role in determining the existence of Kähler-Einstein metrics, that is Kähler metrics whose Ricci curvature is proportional to the metric tensor. They are also significant in determining the convergence of the Kähler-Ricci flow and Kähler-Ricci iteration.

Let V be a smooth Fano G-variety, with G finite.

Definition 51. A Kähler metric $g = g_{i\bar{j}}$ on V with associated Kähler form

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum g_{i\bar{j}} dz_i \wedge d\bar{z_j} \in c_1(V)$$

is *Kähler-Einstein* if $\mathrm{Ricci}(\omega) = \omega$, where $\mathrm{Ricci}(\omega)$ is the Ricci curvature of the metric g.

The following theorem is proved in [DK01], [Nad90], [Tia87] and [CS08, Appendix A].

Theorem 52. *If*

$$\mathrm{lct}\big(V, G\big) > \frac{\dim(V)}{\dim(V)+1},$$

then V admits a G-invariant Kähler-Einstein metric.

The problem of the existence of Kähler-Einstein metrics on smooth del Pezzo surfaces is completely solved by [Tia90b].

Theorem 53. *Let X be a smooth del Pezzo surface, then the following are equivalent:*

- *X admits a Kähler-Einstein metric.*
- *X is not the blowup of $\mathbb{P}^2$ in one or two points.*

- Aut(X) *is reductive.*

Definition 54. The *normalised Kähler-Ricci flow* on V is defined by the following equations:

$$\frac{\partial\omega(t)}{\partial t} = -\mathrm{Ricci}\big(\omega(\mathrm{t})\big) + \omega(\mathrm{t}); \quad \omega(0) = \omega, \tag{4.1}$$

where $\omega(t)$ is a Kähler form on V such that $\omega(t) \in c_1(V)$ and t is a non-negative real number.

Cao proved in [Cao85] that solutions $\omega(t)$ exist for all $t > 0$.

Suppose that V admits a Kähler-Einstein metric with a Kähler form ω_{KE}. The following theorem is due to [Nad90] and [TZ07].

Theorem 55. *Any solution to the normalised Kähler-Ricci flow (4.1) converges to ω_{KE} in the sense of Cheeger-Gromov.*

Let G be a finite subgroup of Aut(V), for smooth Fano manifold V. Suppose now that the initial metric g is G-invariant. This guarantees, by the uniqueness of the solutions to the PDE (4.1) in the definition of the normalised Kähler-Ricci flow, that all the metrics along the flow will also be invariant and so we can reformulate Theorem 55 as follows (cf. [Rub09, San08]).

Theorem 56. *If*

$$\mathrm{lct}\big(V, G\big) > \frac{\dim(V)}{\dim(V)+1}.$$

Then the normalised Kähler-Ricci flow (4.1) converges in the $C^{\infty}(V)$-topology to ω_{KE}.

Definition 57. The *normalised Kähler-Ricci iteration* on V is defined by the equations:

$$\omega_n = \mathrm{Ricci}(\omega_{\mathrm{n+1}}); \quad \omega_0 = \omega \tag{4.2}$$

where ω is a Kähler form such that $\omega \in c_1(V)$.

In Yau's paper [Yau78] he shows that there exist solutions ω_n to the normalised Kähler-Ricci iteration (4.2) for all $n \geqslant 1$.

Starting with a G-invariant Kähler-Einstein metric with associated Kähler form ω_{KE} on a Fano manifold V we have the following condition for smooth convergence due to [Rub07].

Theorem 58. *Any solution to the Kähler-Ricci iteration (4.2) converges to ω_{KE} in the $C^\infty(V)$-topology whenever V is G-strongly-exceptional.*

4.3 Exceptional Quotient Singularities

A *quotient singularity*, $(V \ni P)$ is the quotient of $\mathbb{C}^n$ by a finite subgroup G of $\mathbb{GL}_n(\mathbb{C})$. Let $\mathbb{C}[U]$ be the co-ordinate ring of the variety U. Consider the sub-algebra of invariants of G, $\mathbb{C}[U]^G = \left\{ f \in \mathbb{C}[U] \,\middle|\, gf = f \right\}$, which is finitely generated over $\mathbb{C}$ (see e.g. [SR94, Appendix Section 4]). Thus there exists an affine algebraic variety V such that $\mathbb{C}[V] = \mathbb{C}[U]^G$ and we call this the quotient variety, writing $V = U/G$.

Example 59. By Kawamata ([Kaw88]), we know that quotient singularities in dimension two are precisely Kawamata log terminal (cf. Example 4).

An element $g \in G \leqslant \mathbb{GL}_n(\mathbb{C})$ is called a *quasi-reflection* if we may diagonalise its corresponding matrix to diag$(\epsilon, 1, \ldots, 1)$; for example, the symmetric group $\mathbb{S}_n$ acting on $\mathbb{C}^n$ by permutation of the co-ordinates. There is a famous theorem of the 1950s by Chevalley and Shephard-Todd, which states that the ring of invariants of a finite group acting on a complex vector space is a polynomial ring if and only if the group is generated by quasi-reflections, which is another way of saying that a quotient variety $V = U/G$ is non-singular if and only if G is generated by quasi-reflections. For any G, the quasi-reflections generate a normal subgroup N. By considering the quotient group G/N acting on $\mathbb{C}^n/N \cong \mathbb{C}^n$ we may assume that G contains no quasi-reflections.

Definition 60 (cf. [Sho00, Definition 1.5], [MP99a, Definition 2.5])**.** Let $(P \in V)$ be a normal singularity and let $\Delta = \sum \delta_i \Delta_i$ be a boundary on V, that is Δ is effective and all the coefficients δ_i are less than one. Suppose that (V, Δ) is log canonical, then the pair (V, Δ) is *exceptional* if there exists at most one exceptional divisor E over V with discrepancy -1 with respect to (V, Δ). We say that $(V \ni P)$ is *exceptional* if the pair (V, Δ) is for all possible Δ, where (V, Δ) is log canonical (see [CS09, Thm 3.16] for the equivalence of this with that of Definition 26).

Markushevich and Prokhorov in the articles [MP99b, MP99a, Pro00], give necessary and sufficient conditions in terms of semi-invariants of the group for quotient singularities in dimensions three to be exceptional. This extends a similar result of Shokurov in [Sho00] on two-dimensional quotients.

For a finite subgroup G of $\mathbb{GL}_n(\mathbb{C})$, a function $f \in \mathbb{C}[x_1,\dots,x_n]$ is a *semi-invariant* of G if there exists a homomorphism $\chi : G \longrightarrow \mathbb{C}^*$ such that $g \cdot f = \chi(g) f$ for all $g \in G$. If $\chi = 1$, then f is an *invariant* of G ([Spr77, Definition 4.3.1]).

Theorem 61 (Shokurov [MP99a, Proposition 1.1])**.** *A two-dimensional quotient singularity $V = \mathbb{C}^2/G$ by a finite group without reflections is exceptional if, and only if, G has no semi-invariants of degree less than or equal to two.*

Theorem 62 ([MP99a, Theorem 1.2])**.** *A three-dimensional quotient singularity $V = \mathbb{C}^3/G$ by a finite group without reflections is exceptional if, and only if, G has no semi-invariants of degree less than or equal to three.*

Furthermore, Markushevich and Prokhorov use the Miller-Blichfeldt-Dickson classification of finite subgroups of $\mathbb{GL}_3(\mathbb{C})$ ([BDM16]) to give a complete list of such subgroups yielding an exceptional quotient singularity. Recently, Sakovics in [Sak10] extended these results classifying also those weakly-exceptional $(\mathbb{P}^2, G)$, $G \leqslant \mathbb{PGL}_3(\mathbb{C})$. We present these results in Section 6.9. Observe that exceptionality imposes strong restrictions on the group G.

Definition 63. Let $G \leqslant \mathbb{GL}_n(\mathbb{C})$ be a finite subgroup.

- G is *reducible* (or the action of G on $\mathbb{C}^n$ is *intransitive*), if there exists a proper invariant subspace $W \subset \mathbb{C}^n$, i.e. such that $g \cdot W = W$ for all $g \in G$. Otherwise G is *irreducible* (or we say that the action of G on $\mathbb{C}^n$ is *transitive*, i.e. the action has just one orbit).
- We call G *imprimitive* (of type (m^k)) if there exists a non-trivial decomposition $\mathbb{C}^n = \bigoplus_i^k W_i$ with $\dim(W_i) = m$ such that $g \cdot W_i = W_j$ for any $g \in G$. Otherwise G is *primitive*.
- We have the inclusions $\{\text{primitive}\} \subseteq \{\text{irreducible \& imprimitive}\} \subseteq \{\text{irreducible}\}$.

Let $(V \ni O)$ be the quotient of $(\mathbb{C}^n \ni 0)$ by a finite subgroup $G \leqslant \mathbb{GL}_n(\mathbb{C})$ without reflections.

Proposition 64 ([MP99b, Corollary 2.3])**.** *If $(V \ni O)$ is exceptional, then G is primitive.*

Since it is known that the number of primitive subgroups of $\mathbb{GL}_n(\mathbb{C})$ is limited up to conjugations and additions of scalar matrices, it follows from this that there are only finitely many groups G, up to conjugations and additions of scalar matrices, such that the quotient singularity $\mathbb{C}^n/G$ is exceptional ([Pro00, Theorem 1.1]).

Proposition 65 ([MP99a, Proposition 3.6]). *If G is reducible, then $(V \ni O)$ is not weakly-exceptional.*

Example 66 ([Pro00, Example 3.1]). Let $\Gamma \leqslant \mathbb{SL}_2(\mathbb{C})$ be a binary icosahedral group. Consider the subgroup

$$G = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \,\middle|\, A, B \in \Gamma \right\} \leqslant \mathbb{SL}_4(\mathbb{C}).$$

Then the singularity $\mathbb{C}^4/G$ is non-exceptional as the representation $G \hookrightarrow \mathbb{SL}_4(\mathbb{C})$ is reducible. However, the ring of invariants can be represented in the form

$$\mathbb{C}[x_1, x_2, y_1, y_2]^G \cong \mathbb{C}[x_1, x_2]^\Gamma \otimes \mathbb{C}[y_1, y_2]^\Gamma.$$

Since Γ has no semi-invariants of degree less than or equal to twelve (see [Spr77, 4.5.5]), it follows the same holds true for G. Thus we need to look for something extra to extend Theorem 62 in higher dimensions.

Let $\pi : \mathbb{GL}_{n+1}(\mathbb{C}) \longrightarrow \mathbb{PGL}_n(\mathbb{C}) \cong \mathrm{Aut}(\mathbb{P}^n)$ be the natural projection. We have the following conjecture — a special case of Conjecture 16. Indeed, if there exists a divisor realising the global log canonical threshold then the notions of exceptional and strongly-exceptional coincide.

Conjecture 67 ([CS09, Conjecture 1.23]). *The quotient singularity $\mathbb{C}^{n+1}/G$ is exceptional if, and only if, $\mathrm{lct}\big(\mathbb{P}^n, \pi(G)\big) > 1$. That is, if and only if, the pair $\big(\mathbb{P}^n, \pi(G)\big)$ is strongly-exceptional.*

Lemma 68. *Suppose that $G \leqslant \mathbb{GL}_{n+1}(\mathbb{C})$ has a semi-invariant of degree d, then it follows from the definition that*

$$\mathrm{lct}\big(\mathbb{P}^n, \pi(G)\big) \leqslant \frac{d}{n+1}.$$

Definition 69. Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity, and let $\pi : W \longrightarrow V$ be a birational morphism such that the following hypotheses are satisfied:

- the exceptional locus of π consists of one irreducible divisor $E \subset W$ such that $O \in \pi(E)$,

- the log pair (W,E) has purely log terminal (plt) singularities (see Definition 2),
- the divisor $-E$ is a π-ample $\mathbb{Q}$-Cartier divisor.

Then we say $\pi : W \longrightarrow V$ is a *plt blowup*.

Definition 70. $\big(V \ni O\big)$ is weakly-exceptional if it has unique plt blow up.

Weakly-exceptional Kawamata log terminal singularities do exist (see [Kud01, Example 2.2]).

Theorem 71 ([CS09, Theorem 3.15])**.** *The quotient singularity* $\mathbb{C}^{n+1}/G$ *is weakly-exceptional if, and only if,* $\mathrm{lct}\big(\mathbb{P}^n,\pi(G)\big) \geqslant 1$.

Thus, we see that notions of weakly-exceptional quotient singularity and of weakly-exceptional pairs $\big(\mathbb{P}^n,\pi(G)\big)$ agree here and the corresponding statement to Conjecture 67 is true for weak-exceptionality. Cheltsov and Shramov go further and prove the above conjecture for $\mathbb{P}^2$ with this extension of Theorem 62 and a corresponding statement for weak-exceptionality.

Theorem 72 ([CS09, Theorem 3.17])**.** *The following are equivalent:*

- $\big(V \ni P\big) = \mathbb{C}^3/G$ *is exceptional,*
- *G has no semi-invariants of degree less than or equal to three,*
- $\mathrm{lct}\big(\mathbb{P}^2,\pi(G)\big) \geqslant \frac{4}{3}$.

Theorem 73 ([CS09, Theorem 3.18])**.** *For a three dimensional quotient singularity* $\big(V \ni 0\big)$ *the following are equivalent:*

- *the inequality* $\mathrm{lct}\big(\mathbb{P}^2,\pi(G)\big) \geqslant 1$ *holds,*
- *the group G does not have semi-invariants of degree at most two.*

Cheltsov-Shramov extended Theorem 72 to six-dimensional quotient singularities with the following.

Theorem 74 ([CS10, Thm 1.12])**.** *If* $n \leqslant 5$*, then following are equivalent:*

- $\left(V \ni P\right) = \mathbb{C}^{n+1}/G$ *is exceptional,*

- *G is primitive and has no semi-invariants of degree less than or equal to $n+1$,*

- $\mathrm{lct}\big(\mathbb{P}^n, \pi(G)\big) \geqslant \frac{n+1}{n}$.

To answer completely our Question B for $\mathbb{P}^2$ it remains to identify those finite groups $\pi(G)$ such that $\big(\mathbb{P}^2, \pi(G)\big)$ is weakly-exceptional. That is, we should identify those finite groups $G \leqslant \mathbb{GL}_3(\mathbb{C})$ on the list of Miller-Blichfeldt-Dickson, but not on that of Markushevich and Prokhorov (Proposition 204), who do not have semi-invariants of degree at most two. In fact, this was recently done by Sakovics in [Sak10].

Remark 75. The above discussion shows the link between pairs $\left(\mathbb{P}^2, G\right)$ and exceptional quotient singularities, it would be interesting — but we suspect hard — to understand the link with other del Pezzo pairs $\left(S, G\right)$.

Chapter 5

Preliminaries

We present here notations used in this thesis and collect together the tools for the following Chapter. Again, we re-iterate that throughout, unless otherwise stated, all varieties are assumed to be normal, projective, algebraic and defined over the field of complex numbers.

5.1 Del Pezzo Surfaces

A del Pezzo surface S is an irreducible surface whose anti-canonical divisor is ample — that is a Fano variety of dimension two. We define the *degree* d_S of S to be the self-intersection of the anti-canonical class, i.e. $d_S = K_S^2$ (Definition 19).

5.1.1 Classification of del Pezzo surfaces

Proposition 76 ([Bla06, Proposition 4.1.3]; cf. [Dem80, KSC04, Kol96, Man66])**.** *Let S be a rational surface, then the following are equivalent:*

(i) S is a del Pezzo surface;

(ii) $S \cong \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, or the blow up of $1 \leqslant r = 9 - d_S \leqslant 8$ points of $\mathbb{P}^2$ in general position (that is — no three collinear, no six on the same conic, and no eight lie in a cubic having a double point at one of them);

(iii) $K_S^2 \geqslant 1$ *and any irreducible curve of S has self-intersection* $\geqslant -1$*;*

(iv) $C \cdot (-K_S) > 0$ *for all effective divisors C on S.*

Proof. $(i \Rightarrow iv)$ As $-K_S$ is ample, some multiple of it, $m > 0$ say, is very ample. The map corresponding to $-mK_S$ gives us an embedding; $-mK_S \cdot C$ is the degree of C in this embedding, which is necessarily positive.

$(iv \Rightarrow iii)$ Suppose that some irreducible curve C of S has self-intersection $\leqslant -2$. The adjunction formula ([Bea96, I.15]) gives $C \cdot (C + K_S) = -2 + 2 \cdot p_a(C) \geqslant -2$, whence $C \cdot (-K_S) \leqslant 2 + C^2 \leqslant 0$, which contradicts assertion *(iv)*.

$(iv \Rightarrow ii)$ S is a rational surface such that *(iv)* and *(iii)* hold. Applying the MMP to S by blowing down some (-1)-curves on S, we get a birational morphism to a surface T isomorphic to the projective plane $\mathbb{P}^2$ or to some Hirzebruch surface $\mathbb{F}_n$ with $n \neq 1$ (see Proposition 17). Observe that no curve on T has self-intersection less than -1 as no curve on S does. As any Hirzebruch surface $\mathbb{F}_n$ has a unique curve of self-intersection $-n$, $n \leqslant 1$ — that is $T \cong \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. If $S = T$ then we are done, otherwise without loss of generality (since the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the blowup of $\mathbb{P}^2$ in two distinct points) we may suppose that $T \cong \mathbb{P}^2$. Thus, S is the blowup of the projective plane in some number of points in general position and all of these points lie in $\mathbb{P}^2$. Indeed, if any of the points are infinitely near or not in general position then there would exist a curve on S with self-intersection less than -1. To show *(iv)* $\Rightarrow$ *(ii)*; suppose that S is the blowup of $\mathbb{P}^2$ in nine or more points, then there exist conics on $\mathbb{P}^2$ passing through any nine of these points, which are all irreducible as the points are in general position. Take one of them and observe that its strict transform on S intersects the anti-canonical divisor of S non-positively.

$(iii \Rightarrow ii)$ Using the above proof of $(iv \Rightarrow ii)$, we see that S is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the blowup of $\mathbb{P}^2$ in one or more points in general position. Using the formula for the blowup, it is easy to see that $K_S^2 \geqslant 1$ implies that the number of points must be less than 9.

$(ii \Rightarrow i)$ In [Dem80], Theorem 1, it is proved that the blow-up of $1 \leqslant r \leqslant 8$ points in general position gives a del Pezzo surface. The cases of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are clear.

□

Proposition 77 (Anti-canonical Models [KSC04, Theorem 3.36])**.** *Let S be a del Pezzo surface. For:*

(i) $K_S^2 = 1$, *S is isomorphic to a hypersurface of degree six in the weighted projective space* $\mathbb{P}(1,1,2,3)$.

(ii) $K_S^2 = 2$, *S is isomorphic to a hypersurface of degree four in the weighted projective space* $\mathbb{P}(1,1,1,2)$.

(iii) $K_S^2 = 3$, *S is isomorphic to a cubic surface in* $\mathbb{P}^3$.

(iv) $K_S^2 = 4$, *S is isomorphic to the complete intersection of two quadrics in* $\mathbb{P}^4$.

Further details can be found in, for example, [Dem80, KSC04, Kol96, Man66].

5.1.2 Lines on del Pezzo surfaces

Clearly the Picard group (the group of all divisors modulo numerical equivalence) on $\mathbb{P}^2$ is generated by the class of a general line L. For $\pi : S_d \longrightarrow \mathbb{P}^2$, the blowup of $\mathbb{P}^2$ in $r = 9-d$ points $P_1, \ldots, P_r$ in general position, S_d is a del Pezzo surface of degree d as seen above. It is easy to see that the Picard group of S_d, $\mathrm{Pic(S_d)}$, is generated by the strict transform of L and the $9-d$ exceptional curves $E_1, \ldots, E_r$ where $\pi(E_r) = P_r$. Furthermore, the anti-canonical class can be written as $-K_S = -3\pi^*(L) + \sum_{r=1}^{9-d} E_r$.

For $d > 2$, the anti-canonical map

$$\varphi_{|-K_S|} : S_d \hookrightarrow \mathbb{P}^d$$

embeds S_d in $\mathbb{P}^d$ — hence the image of a (-1)-curve is a line. Thus, it is natural to refer to (-1)-curves on S_d as *lines* (for $d = 1,2$ we do not call (-1)-curves lines). To enumerate the number of these lines or (-1)-curves on S_d is relatively simple process — they are the exceptional curves $E_1, \ldots, E_r$ and the strict transforms of the curves of degree δ passing through the $P_1, \ldots, P_r$.

For $9-d \leqslant 4$, that is for $d \geqslant 6$, the (-1)-curves are given by the $9-d$ exceptional divisors

$E_1, \ldots, E_r$ and the strict transforms of the lines $\langle P_i P_j \rangle$ between the blowup points. This gives

$$(9-d) + \binom{9-d}{2} \tag{5.1}$$

lines on S_d. For $9-d \geqslant 5$, we have in addition to (5.1), the strict transform of the $\binom{9-d}{5}$ conics passing through five points on $\mathbb{P}^2$. When $9-d=7$, we add in the seven cubics with a node at some point P_k. For $9-d=8$; we add $2\binom{8}{6}$ cubics with a node at some point, $\binom{8}{3}$ quartics with nodes at three distinct points, $\binom{8}{2}$ quintics with nodes at all but two points and eight sextics with nodes at all points except one where the sextic has a multiplicity three. This can be summed up in Tables 5.1 and 5.2, the first of which we take from [Bla06, Proposition 4.2.2].

$9-d$	degree δ	multiplicities at the points	number of such curves for $9-d=1$,	2,	3,	4,	5,	6,	7,	8
$\geqslant 2$	1	$(1,1)$		1	3	6	10	15	21	28
$\geqslant 5$	2	$(1,1,1,1,1)$					1	6	21	56
$\geqslant 7$	3	$(2,1,1,1,1,1,1)$							7	56
8	4	$(2,2,2,1,1,1,1,1)$								56
8	5	$(2,2,2,2,2,2,1,1)$								28
8	6	$(3,2,2,2,2,2,2,2)$								8

Table 5.1: Lines on del Pezzo surfaces.

degree of S_d, d	9	8	7	6	5	4	3	2	1
$9-d$	0	1	2	3	4	5	6	7	8
no. of (-1)-curves	0	1	3	6	10	16	27	56	240

Table 5.2: Number of (-1)-curves on del Pezzo surfaces S_d of degree d.

5.1.3 Automorphism groups of del Pezzo surfaces

As our main objects of study here are smooth del Pezzo G-surfaces S, we should say something about the possibilities for the group G that acts regularly on S. Let $\pi : S_d \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ in the points $P_1, \ldots, P_r$ in general position for $0 \leqslant r \leqslant 8$, then S_d is a del Pezzo surface of degree $d = 9-r$. Our first observation is that Aut(S) acts on the Picard lattice $\mathrm{Pic(S)} \cong \mathbb{Z}^{\mathrm{rkPic(S)}}$ generated by the strict transform of the class of a line on $\mathbb{P}^2$ and the exceptional divisors $E_1, \ldots, E_r$ — sending in particular (-1)-curves to (-1)-curves. If $r \geqslant 4$, then it follows that Aut(S) is finite. For $r \leqslant 3$ there are, in addition, projective transformations of $\mathbb{P}^2$ permuting the points $P_1, \ldots, P_r$ that lift to S_d that do not occur for $r \geqslant 4$ as all quadrilaterals are projectively similar in $\mathbb{P}^2$. Thus for $r \geqslant 4$, that is for $d \leqslant 5$, we have the following Lemma.

Lemma 78. *Let S_d be a smooth del Pezzo surface of degree d then for $d \leqslant 5$, $\mathrm{Aut}(S_d)$ is finite.*

For larger degree del Pezzo surfaces we discuss their automorphism groups at the beginning of the relevant sections.

In order to answer our Question C, we need first a classification of the possible finite automorphism groups G of smooth del Pezzo G-surfaces such that the pair (S, G) is minimal. This work was completed in [DI10] (see their introduction for an account of the history of the problem), from which we take the list of possibilities for G. Also worth mentioning are the works [Hos96, Hos02, Hos97] and [Koi88]. The papers of Hosoh examine computationally the automorphism groups of cubic and quartic del Pezzo surfaces. In [Koi88], Koitabashi gives the automorphism group of a generic rational surface of rank $r+1$ — that is, rational surfaces S that are the blowups of $\mathbb{P}^2$ in r points in general position. From Proposition 76, if $r \leqslant 8$ then S is a del Pezzo surface. We include the main result of [Koi88] below.

Proposition 79 ([Koi88])**.** *Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic and S the blow-up of $\mathbb{P}^2$ in points $P_1, \ldots, P_r$ in general position. Then the group $\mathrm{Aut}_{\Bbbk}(S)$ is given in Table 5.3, where the notation $\mathbb{PGL}_3(\Bbbk; P_1, \ldots, P_i)$ denotes the subgroup of $\mathbb{PGL}_3(\Bbbk)$ that fixes the points $P_1, \ldots, P_i$.*

r	0	1	2	3
$\mathrm{Aut}_{\Bbbk}(S)$	$\mathbb{PGL}_3(\Bbbk)$	$\mathbb{PGL}_3(\Bbbk; P_1)$	$\mathbb{PGL}_3(\Bbbk; P_1, P_2)$	$\mathbb{PGL}_3(\Bbbk; P_1, P_2, P_3) \rtimes \mathbb{Z}_2$

r	4	5	6	7	8	$9 \leqslant r$
$\mathrm{Aut}_{\Bbbk}(S)$	$\mathbb{S}_5$	$\mathbb{Z}_2^4$	$\{\mathrm{id}\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{\mathrm{id}\}$

Table 5.3: Automorphism groups of generic rational surfaces of rank $r+1$.

5.1.4 Del Pezzo surfaces of degrees one and two

As we saw above in Proposition 79, del Pezzo surfaces of degrees one and two have natural involutions defined on them — these are the Bertini and Geiser involutions, respectively.

From Proposition 77, we see that there are natural maps associated with the respective anti-canonical linear systems

$$\mathbb{P}(1,1,2,3) \supset S_1 \xrightarrow{2:1} \mathbb{P}(1,1,2)$$

and

$$\mathbb{P}(1,1,1,2) \supset S_2 \xrightarrow{2:1} \mathbb{P}^2,$$

where S_d is a del Pezzo surface of degree d. The Bertini involution on S_1 and the Geiser involution on S_2 can be viewed as the involutions associated with the interchanging of the sheets of the double cover of $\mathbb{P}(1,1,2)$ or $\mathbb{P}^2$ respectively.

Remark 80. Let S be a smooth del Pezzo surface of degree one or two and let τ be the Bertini or Geiser involution, respectively. Then since curves in $|-K_S|$ are invariant under τ, for any effective $\mathbb{Q}$-divisor $\Delta \equiv -K_S$ on S where $\mathrm{LCS}\big(\mathrm{S},\Delta\big)$ is zero-dimensional, the points of $\mathrm{LCS}\big(\mathrm{S},\Delta\big)$ are fixed under the action of τ. That is points of $\mathrm{LCS}\big(\mathrm{S},\Delta\big)$ belong to the ramification curve of the double cover.

In Section 6.1, we require the following technical results on the Bertini involution. Let S_1 be a smooth del Pezzo surface of degree one with automorphism group G, C an element of $|-K_{S_1}|$. Let

$$\sigma : X \longrightarrow S_1$$

be the blowup of S_1 at the point $P \in C$ with exceptional divisor over P, E. Let β be the Bertini involution lifted to S_1 and write $\bar{C}$, $\bar{P}$, etc. for the strict transforms of C and P, respectively.

Lemma 81. *The action of $\beta \in G$ lifted upstairs onto X acts on E with exactly two fixed points.*

Proof. Let x,y,z,t be weighted homogeneous coordinates of weights $1,1,2,3$, respectively. Locally on $S_1 \subset \mathbb{P}_{x,y,z,t}(1,1,2,3)$, we may write the equation of the curve C as $t^2 = x$. On one chart of the blow-up the pullback of this curve is given by

$$\sigma^*\Big(\big\{t^2 - x = 0\big\}\Big) = \big\{t' = x'\big\}$$

(σ^* sends $t \mapsto t'$ and $x \mapsto t'x'$). Now, writing $\bar{\beta}$ for the involution β lifted on X, β maps $t \mapsto -t$ and so $\bar{\beta}$ sends $t' \mapsto -t'$ and $x' \mapsto -x'$. Thus there are exactly two fixed points $(0,1)$ and

(1,0). □

Lemma 82. *For $\lambda \in \mathbb{Q}$, let $D \equiv -K_{S_1}$ be a G-invariant effective $\mathbb{Q}$-divisor. Suppose that* $\mathrm{LCS}(\mathrm{S}_1, \lambda\mathrm{D})$ *is zero-dimensional and* $(S_1, \lambda D)$ *is not log canonical at the point* $P \in S_1$*. Then there exists a point* $Q \in E$ *such that* $(X, \lambda\bar{D} + (\mathrm{mult}_P \lambda D - 1)E)$ *is not log canonical at* Q *and there are less than* λ *points in the* G*-orbit of* Q*.*

Proof. By taking the log pull-back of the pair $(S_1, \lambda D)$, we see that the pair

$$\left(X, \lambda\bar{D} + (\mathrm{mult}_p \lambda D - 1)E\right)$$

is not log canonical at some point Q. By adjunction, it follows that the pair $(E, \lambda\bar{D}|_E)$ is not log canonical at the point Q. Hence $\mathrm{mult}_Q \lambda\bar{D}|_E > 1$. Moreover, the following inequality holds

$$\sum \mathrm{mult}_{Q_j} \lambda\bar{D}|_E > J$$

where the sum is taken over all J points Q_j that form the G-orbit of $Q = Q_0$. Since $\bar{D}|_E = (\mathrm{mult}_P D)\bar{P}$ and $\mathrm{mult}_P D \leqslant 1$ we have that $J < \lambda \sum \mathrm{mult}_{Q_j} \bar{P}$, which yields the required result on noting that $\sum \mathrm{mult}_{Q_j} \bar{P}$ is zero when $Q_j \neq \bar{P}$ and one otherwise. □

5.1.5 Some results on cubic surfaces

The following two results we use later in some calculations on cubic surfaces (Section 6.3.3). Let S_3 be a smooth del Pezzo surface of degree three, that is, a smooth cubic surface.

Definition 83. An *Eckardt point* is a point on S_3 where three lines intersect.

Remark 84. A cubic surface may have up to a maximum of eighteen Eckardt points.

Proposition 85 ([Dol10, Cubic Surfaces Chapter])**.** *There is a bijective correspondence between Eckardt points on* S_3 *and automorphisms of order 2 with one isolated fixed point.*

Proof. [⇒] Let $P \in S_3$ be an Eckardt point, with lines $L_1, L_2, L_3 \ni P$ and consider the blowup $\pi : S_2 \longrightarrow S_3$ at the point P depicted in Figure 5.1, with exceptional divisor E. The surface S_2

is del Pezzo of degree two and the pre-image of $|-K_{S_3}-P|$ is the anti-canonical linear system $|-K_{S_2}|$. The strict transforms $\overline{L_i}$ of the (-1)-curves L_i are of course (-2)-curves on S_2.

The anti-canonical map $f_{|-K_{S_2}|}: S_2 \longrightarrow \mathbb{P}^2$ is a double cover of the projective plane with ramification divisor R. The images $f(\overline{L_i})$ of the strict transforms of the lines L_i must be singular points of the ramification divisor R (abusing notation we write R for both the ramification divisor on S_2 and its image on $\mathbb{P}^2$) as the $\overline{L_i}$ are (-2)-curves. The image of the exceptional curve $f(E)$ is a line on $\mathbb{P}^2$ passing through three singular points of R — a quartic curve. Thus, $f(E)$ is an irreducible component of R and hence $R = f(E) + C$ where C is an irreducible cubic intersecting $f(E)$ at the three singular points A_1, A_2, A_3.

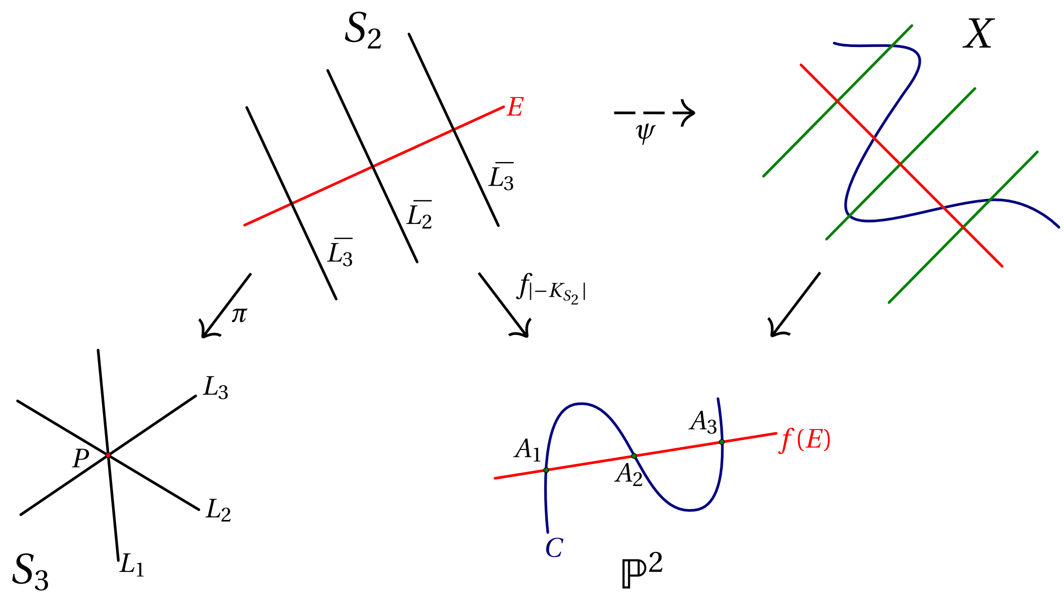


Figure 5.1: Eckardt points on a cubic and involutions with one isolated fixed point.

Let X be the double cover of the blowup of $\mathbb{P}^2$ at the three points A_k, ramified along the strict transform of the curve R. Then there exists a birational map $\psi: S_2 \dashrightarrow X$ which is regular outside of the locus $\cup_i^3 \overline{L_i}$. We may extend ψ to a map regular on the whole of S_2 by mapping the $\overline{L_i}$ to the pre-images of the points A_k under the map $f: S_2 \longrightarrow \mathbb{P}^2$. It follows that $\psi: S_2 \longrightarrow X$ is a finite map of degree two and hence a Galois cover. Corresponding to this Galois cover there is an automorphism of S_2 (involution) that leaves E point-wise invariant — thus descending to an involution, g, on S_3. Since it leaves $|-K_{S_3}|$ invariant, it must be induced by a linear projective transformation $\bar{g}$ of $\mathbb{P}^3$. The fixed points of g in $\mathbb{P}^3$ is the point P and a plane Π such that $\Pi \cap S_3 = \widehat{C}$, where the linear projection from P maps $\widehat{C}$

isomorphically to C.

[$\Leftarrow$]Suppose that g is an involution of S_3 with one isolated fixed point. Then g is induced by an automorphism $\bar{g}$ of $\mathbb{P}^3$. We diagonalise the action of $\bar{g}$ on $\mathbb{C}^4$ and find that its Eigenspace has Eigen-subspaces of dimensions one and three. It follows that $\bar{g}$ fixes a point P and a plane Π, on S_3 g fixes the point P and a plane cubic curve, $\Pi \cap S_3 = \widehat{C} \ni P$. Let Σ be the tangent plane to S_3 at the point P. Then Σ is invariant and its intersection with S_3 is a plane cubic curve $\Sigma \cap S_3 = D$. As both D and $\widehat{C}$ are numerically equivalent to the anti-canonical divisor they intersect in three smooth points, which are fixed as $\widehat{C}$ is. The only possibility is for D to be the sum of three lines, hence P is an Eckardt point. Indeed, if D is irreducible then its normalisation is isomorphic to $\mathbb{P}^1$ which only has two fixed points for any involution. If D is the product of a line and a conic then one component must have three fixed points including P which is impossible as before. □

Proposition 86 ([Dol10, Cubic Surfaces Chapter])**.** *No more than two Eckardt points lie on a line contained in the cubic surface S_3.*

Proof. Let S_3 be a smooth cubic surface with two Eckardt points P_1, P_2 that lie on some line L and consider the linear projection

$$\varphi : S_3 \xrightarrow{2:1} \mathbb{P}^2$$

from the point P_1. The ramification curve consists of the line L and a cubic curve C, where C is the locus of all points on S_3 such that the lines joining them to P_1 are tangent to S_3. Writing Q for $\varphi(P_2)$, it is easy to see that Q belongs to $\varphi(L) \cap \varphi(C)$. The point P_2 belongs to the fixed curve of the involution g corresponding to the Eckardt point P_1 and the image of the plane tri-tangent to S_3 at P_2 is a line on $\mathbb{P}^2$ intersecting $\varphi(C)$ at only one point. Thus, the point Q must be a point of inflection for $\varphi(C)$ and it follows that there cannot be another Eckardt point on the line L. □

5.2 Nadel Vanishing and Corollaries

5.2.1 Locus of log canonical singularities and multiplier ideals

For the convenience of the reader, we include several definitions from [CS08, Section 2]. Further details on these can be found there, or in [Laz04, Book II]. Let V be a $\mathbb{Q}$-factorial variety with no worse than klt singularities and consider an effective $\mathbb{Q}$-divisor $\Delta_V = \sum_{i=1}^{r} \delta_i \Delta_i$, where the Δ_i are distinct prime Weil divisors on the variety V and let $\pi : U \longrightarrow V$ be a birational morphism, such that U is smooth. Write

$$\Delta_U = \sum_{i=1}^{r} \delta_i \bar{\Delta}_i,$$

where $\bar{\Delta}_i$ is a proper transform of the divisor Δ_i on the variety U. Then

$$K_U + \Delta_U \sim_{\mathbb{Q}} \pi^*\Big(K_V + \Delta_V\Big) + \sum_{i=1}^{n} a_i E_i,$$

where $a_i = a(E_i; V, \Delta) \in \mathbb{Q}$ is the discrepancy of E_i with respect to (V, Δ), and E_i is an exceptional divisor of the morphism π. Suppose that

$$\left(\bigcup_{i=1}^{r} \bar{\Delta}_i\right) \bigcup \left(\bigcup_{i=1}^{n} E_i\right)$$

is a divisor with simple normal crossing — that is π is a log resolution of (V, Δ).

Recall (Definition 2) that the singularities of (V, Δ_V) are log canonical (resp., log terminal) if

- the inequality $\delta_i \leqslant 1$ holds (resp., the inequality $\delta_i < 1$ holds),
- the inequality $a_j \geqslant -1$ holds (resp., the inequality $a_j > -1$ holds),

for every $i = 1, \ldots, r$ and $j = 1, \ldots, n$.

Definition 87 ([CS08, Definition 2.1])**.** The *locus of log canonical singularities* of the log pair (V, Δ_V) is the set

$$\mathrm{LCS}\Big(\mathrm{V}, \Delta_{\mathrm{V}}\Big) = \left(\bigcup_{\delta_i \geqslant 1} \Delta_{\mathrm{i}}\right) \bigcup \left(\bigcup_{a_i \leqslant -1} \pi\big(\mathrm{E_i}\big)\right) \subsetneq \mathrm{V}$$

Definition 88 ([CS08, Definition 2.2])**.** A proper irreducible sub-variety $W \subsetneq V$ is said to be a *centre of log canonical singularities* of the log pair (V, Δ_V) if one of the following conditions is satisfied:

- either the inequality $\delta_i \geqslant 1$ holds and $W = \Delta_i$,
- or the inequality $a_i \leqslant -1$ holds and $W = \pi(E_i)$ for some choice of the birational morphism $\pi : U \longrightarrow V$.

Let $\mathbb{LCS}(V, \Delta_V)$ be the set of all centres of log canonical singularities of (V, Δ_V). Then

$$W \in \mathbb{LCS}\Big(V, \Delta_V\Big) \Longrightarrow W \subseteq \mathrm{LCS}\Big(\mathrm{V}, \Delta_{\mathrm{V}}\Big)$$

and $\mathbb{LCS}(V, \Delta_V) = \varnothing \Longleftrightarrow \mathrm{LCS}(\mathrm{V}, \Delta_{\mathrm{V}}) = \varnothing \Longleftrightarrow$ the log pair (V, Δ_V) is log terminal.

Lemma 89 ([CS08, Remark 2.3])**.** *Let X be a variety with log terminal singularities with effective $\mathbb{Q}$-divisor Δ, let $\mathcal{H}$ be a linear system on X with no base points and let $H \in \mathcal{H}$ be a sufficiently general divisor. For a proper irreducible sub-variety $Y \subsetneq X$, with*

$$Y|_H = \sum_{i=1}^{m} Z_i$$

where the $Z_i \subset H$ are irreducible subvarieties, it follows from the definition of $\mathbb{LCS}$ (Definition 88) that

$$Y \in \mathbb{LCS}(X, \Delta) \Longleftrightarrow \{Z_1, \ldots, Z_m\} \in \mathbb{LCS}(H, \Delta|_H).$$

The locus $\mathrm{LCS}(\mathrm{V}, \Delta_{\mathrm{V}}) \subset \mathrm{V}$ can be equipped with a scheme structure (see [Nad90], [Sho93]). Put

$$\mathcal{I}\Big(V, \Delta_V\Big) = \pi_* \mathcal{O}_U\Big(\sum_{i=1}^{n} \lceil a_i \rceil E_i - \sum_{i=1}^{r} \lfloor \delta_i \rfloor \bar{\Delta}_i\Big),$$

and let $\mathcal{L}(V, \Delta_V)$ be a sub-scheme that corresponds to the ideal sheaf $\mathcal{I}(V, \Delta_V)$.

Definition 90 ([CS08, Definition 2.5]; cf. [Laz04, Definition 9.2.1])**.** For the log pair (V, Δ_V), we say that

- the sub-scheme $\mathcal{L}(V, \Delta_V)$ is the sub-scheme of log canonical singularities of (V, Δ_V),
- the ideal sheaf $\mathcal{I}(V, \Delta_V)$ is the multiplier ideal sheaf of (V, Δ_V).

It follows from the construction of the sub-scheme $\mathcal{L}(V, \Delta_V)$ that

$$\mathrm{Supp}\Big(\mathcal{L}(V, \Delta_V)\Big) = \mathrm{LCS}(V, \Delta_V) \subset V.$$

5.2.2 Nadel vanishing

Theorem 91 (Nadel Vanishing). *Let V be smooth projective variety; Δ, H $\mathbb{Q}$-divisors on V with H nef and big and L a $\mathbb{Z}$-divisor on V such that $L \sim_{\mathbb{Q}} K_V + \Delta + H$. Then*

$$H^i\Big(V, \mathcal{O}_V(L) \otimes \mathcal{J}(V, \Delta)\Big) = 0 \quad \text{for} \quad i > 0.$$

Proof. See [Laz04, Theorem 9.4.8]. □

Theorem 92 (Shokurov Connectedness). *Let V be smooth projective variety; Δ, H $\mathbb{Q}$-divisors on V with H nef and big and L a $\mathbb{Z}$-divisor on V such that $L \sim_{\mathbb{Q}} K_V + \Delta + H$.*

Suppose that $-(K_V + \Delta)$ is nef and big, then $\mathrm{LCS}(\mathrm{V}, \Delta)$ is connected.

Proof. Take $L = 0$, then we have the following exact sequence

$$0 \longrightarrow \mathcal{J}(V, \Delta) \otimes \mathcal{O}_V \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_{\mathcal{L}(V,\Delta)} \longrightarrow 0$$

where $\mathcal{L}(V, \Delta)$ is the sub-scheme associated with the multiplier ideal sheaf $\mathcal{J}(V, \Delta)$. The corresponding exact sequence of cohomology groups is

$$\mathbb{C} \cong H^0(V, \mathcal{O}_V) \longrightarrow H^0\Big(\mathcal{L}(V, \Delta), \mathcal{O}_{\mathcal{L}(V,\Delta)}\Big) \longrightarrow H^1\Big(V, \mathcal{J}(V, \Delta) \otimes \mathcal{O}_V)\Big) = 0$$

Thus, $\mathrm{LCS}(\mathrm{V}, \Delta) = \mathrm{Supp}\Big(\mathcal{L}(\mathrm{V}, \Delta)\Big)$ is connected. □

Corollary 93. *Let V be smooth projective variety; Δ, H $\mathbb{Q}$-divisors on V with H nef and big and L a $\mathbb{Z}$-divisor on V such that $L \sim_{\mathbb{Q}} K_V + \Delta + H$.*

Suppose that $\mathrm{LCS}(\mathrm{V}, \Delta)$ is zero dimensional. Then $\mathrm{LCS}(\mathrm{V}, \Delta)$ consists of at most k points, where k is the dimension of $H^0(V, \mathcal{O}_V(L))$.

Proof. By Nadel Vanishing (Theorem 91), $H^i\Big(V, \mathcal{O}_V(L)\otimes \mathcal{J}\big(V,\Delta\big)\Big) = 0$ for $i > 0$. We have the following short exact sequence

$$0 \longrightarrow \mathcal{J}\big(V,\Delta\big)\otimes \mathcal{O}_V(L) \longrightarrow \mathcal{O}_V(L) \longrightarrow \mathcal{O}_{\mathcal{L}(V,\Delta)}(L) \longrightarrow 0$$

where $\mathcal{L}\big(V,\Delta\big)$ is the sub-scheme associated with the multiplier ideal sheaf $\mathcal{J}\big(V,\Delta\big)$. Together these yield the following exact sequence

$$\mathbb{C}^k \cong H^0\big(V, \mathcal{O}_V(L)\big) \longrightarrow H^0\Big(\mathcal{L}\big(V,\Delta\big), \mathcal{O}_{\mathcal{L}(V,\Delta)}(L)\Big) \longrightarrow H^1\Big(V, \mathcal{J}\big(V,\Delta\big)\otimes \mathcal{O}_V(L)\Big) = 0$$

That is to say, the following map is surjective

$$\mathbb{C}^k \longrightarrow H^0\Big(\mathcal{L}\big(V,\Delta\big), \mathcal{O}_{\mathcal{L}(V,\Delta)}(L)\Big)$$

Hence, $\mathrm{LCS}\big(\mathrm{V},\Delta\big) = \mathrm{Supp}\Big(\mathcal{L}\big(\mathrm{V},\Delta\big)\Big)$ consists of at most k points. □

5.2.3 Corollaries on del Pezzo surfaces

Corollary 94. *Let X be a smooth del Pezzo surface, $\lambda \geqslant 0$, $D \sim_{\mathbb{Q}} -K_X$ an effective $\mathbb{Q}$-divisor on X such that $\big(X,\lambda D\big)$ is not klt and* $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ *is zero dimensional. Then the points of* $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ *impose independent linear conditions on elements in $H^0\Big(X, \mathcal{O}_X\big(-(\ulcorner\lambda-1\urcorner)K_X\big)\Big)$.*

Proof. Let $H = (\lambda - 1 - \ulcorner\lambda-1\urcorner)K_X$ and $L = K_X + \lambda D + H \sim_{\mathbb{Q}} -(\ulcorner\lambda-1\urcorner)K_X$. Then H is nef and big, and L is a $\mathbb{Z}$-div on X. Applying Nadel vanishing (Theorem 91) with this H and L shows that the following map is surjective

$$H^0\Big(X, \mathcal{O}_X\big(-(\ulcorner\lambda-1\urcorner)K_X\big)\Big) \longrightarrow H^0\Big(\mathcal{L}\big(X,\lambda D\big), \mathcal{O}_{\mathcal{L}(X,\lambda D)}\big(-(\ulcorner\lambda-1\urcorner)K_X\big)\Big).$$ □

Examples 95. With the assumptions of Corollary 94, suppose also that

(i) $\lambda < 1$. Then we have the surjective map

$$\mathbb{C} \cong H^0\big(X, \mathcal{O}_X\big) \longrightarrow H^0\Big(\mathcal{L}\big(X,\lambda D\big), \mathcal{O}_{\mathcal{L}(X,\lambda D)}\Big)$$

which implies that $\mathrm{LCS}(\mathrm{X}, \lambda\mathrm{D})$ consists of at most one point.

(ii) $\lambda < 2$. Then we have the surjective map

$$\mathbb{C}^{K_X^2+1} \cong H^0\big(X, \mathcal{O}_X(-K_X)\big) \longrightarrow H^0\Big(\mathcal{L}\big(X, \lambda D\big), \mathcal{O}_{\mathcal{L}(X,\lambda D)}(-K_X)\Big)$$

which implies that $\mathrm{LCS}(\mathrm{X}, \lambda\mathrm{D})$ consists of at most $K_X^2 + 1$ points. Furthermore, the surjectivity of the map also implies that for each point Q in $\mathrm{LCS}(\mathrm{X}, \lambda\mathrm{D})$ we may find a curve in $|-K_X|$ that passes through all other points of $\mathrm{LCS}(\mathrm{X}, \lambda\mathrm{D})$ except Q.

Lemma 96. *Let X be a minimal del Pezzo G-surface of degree n and let $\lambda < \xi$, where ξ is the smallest integer such that $|-\xi K_X|^G$ is non-empty. Suppose that there exists a G-invariant effective $\mathbb{Q}$-divisor $D = \sum_{i=0}^r d_i D_i \equiv -K_X$ such that $\big(X, \lambda D\big)$ is not log canonical. Then* $\mathrm{LCS}(\mathrm{X}, \lambda\mathrm{D})$ *is zero-dimensional.*

Proof. Suppose that $\mathrm{LCS}(\mathrm{X}, \lambda\mathrm{D})$ is not zero-dimensional. Then there exist d_k such that $\lambda d_k > 1$. Writing $D = d_k\big(\Delta_1 + \cdots + \Delta_k\big) + \Omega$, where $\Delta_1 + \cdots + \Delta_k$ is a G-orbit of $D_k = \Delta_k$. By Proposition 18, clearly $\Delta_1 + \cdots + \Delta_k \in |-\mu K_X|^G$ for some $\mu \in \mathbb{Z}_{>0}$. However, intersecting λD with $-K_X$ leads to a contradiction.

$$\lambda n = \lambda D \cdot (-K_X) \geqslant \mu\lambda d_k n > \mu n$$

That is, $\xi > \lambda > \mu$ which implies that $|-\mu K_X|^G = \emptyset$. □

Theorem 97 ([Che08, Lemma 5.1])**.** *Let X be a smooth del Pezzo surface, H a Cartier divisor on X, G be a finite sub-group of* $\mathrm{Aut}(\mathrm{X})$ *such that* $\mathrm{Pic}^{\mathrm{G}}(\mathrm{X}) = \mathbb{Z}$ *and* $\mathrm{Pic}^{\mathrm{G}}(\mathrm{X})$ *is generated by H. Furthermore, let ξ be the smallest integer such that $|\xi H| \neq \emptyset$, k the smallest integer such that $k = |\Sigma|$ where Σ is a G-orbit on X and let r be the biggest integer such that $-K_X \sim rH$.*

Then

$$h^0\Big(X, \mathcal{O}_X\big((\xi - r)H\big)\Big) < k$$

implies that

$$\mathrm{lct}\big(X, G\big) = \frac{\xi}{r}$$

Proof. Firstly observe that, by definition $\operatorname{lct}(X,G) \leqslant \frac{\xi}{r}$. Suppose, for a contradiction, that $\operatorname{lct}(X,G) < \frac{\xi}{r}$. Then there exists $\lambda \in \mathbb{Q}_{>0}$ with $\lambda < \frac{\xi}{r}$ and a G-invariant effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X, \lambda D)$ is not log canonical.

By the proof of Lemma 96, $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})$ is zero-dimensional. With notation as above and writing $\mathcal{L}$ for $\mathcal{L}(V,\Delta)$, it follows from Theorem 91 (Nadel Vanishing) that the sequence

$$H^0\Big(X, \mathcal{O}_X\big((\xi-r)H\big)\Big) \to H^0\Big(\mathcal{L}, \mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_X\big((\xi-r)H\big)\Big) \to 0$$

is exact. However this, in conjunction with the G-invariance of $\mathcal{L}(X,\lambda D)$, implies that we have the following contradictory inequality:

$$\begin{aligned} k > h^0\Big(X, \mathcal{O}_X\big((\xi-r)H\big)\Big) &\geqslant h^0\Big(\mathcal{L}, \mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_X\big((\xi-r)H\big)\Big) \\ &= h^0(\mathcal{L},\mathcal{L}) \geqslant \Big|\mathrm{Supp}(\mathcal{L})\Big| = \Big|\mathrm{LCS}(\mathrm{X},\lambda\mathrm{D})\Big| \geqslant \mathrm{k}. \end{aligned}$$

□

5.3 Group Theory Notation

Recall the following definitions. Let A and B be groups, then

- $A \times B$ is the *direct product* of A and B — defined to be the set of ordered pairs (a,b) $(a \in A,\ b \in B)$ with $(a,b)(a',b') = (aa', bb')$;
- $A{\bullet}B$ (or AB) is the *upward extension* of A by B, or the *downward extension* of B by A — defined as the group with normal subgroup A, such that the corresponding quotient group has structure B;
- $A \rtimes B$ (or $A:B$ in [CCN+84]) is the *semi-direct product* of A and B — defined as special case of $A{\bullet}B$ as follows: Given the homomorphism $\varphi : B \longrightarrow \mathrm{Aut(A)}$ that shows how B acts by conjugation on A, we define $A \rtimes B$ to be all the pairs (b,a) $(b \in B, a \in A)$ with $(b,a)(b',a') = (bb', a^{\varphi(b')}a')$. As a memory aid the symbol $\rtimes$ should remind the reader of the symbol used to denote normal subgroups, as we have $A \trianglelefteq A \rtimes B$. Note that $A \rtimes B$ is a split extension $A{\bullet}B$;

- $A^{\bullet}B$ is a non-split extension $A_{\bullet}B$.

Definition 98. For a group G, we say elements $a, b \in G$ are *conjugate* whenever there exists an element $\gamma \in G$ such that $\gamma^{-1}a\gamma = b$. Conjugacy is an equivalence relation and partitions G into a number of distinct conjugacy classes. For subgroups E and F of G, E is *conjugate* to F in G if there exists $\eta \in G$ such that $\eta^{-1}E\eta = F$.

We use the following group theory notation:

Write

$$\left.\begin{array}{r} \mathbb{S}_n \\ \mathbb{A}_n \\ \mathbb{D}_{2n} \\ \mathbb{Z}_n \\ \mathbb{GL}_n(\Bbbk) \\ \mathbb{SL}_n(\Bbbk) \\ \mathbb{PGL}_n(\Bbbk) \\ \mathbb{PSL}_n(\Bbbk) \\ \mathrm{Aut(X)} \\ \mathrm{Bir(X)} \\ \mathrm{Cr_n}(\Bbbk) \\ \mathrm{Pic(X)} \end{array}\right\} \text{for the} \left\{\begin{array}{l} \text{symmetric group on } n \text{ letters;} \\ \text{alternating group;} \\ \text{dihedral group of order } 2n; \\ \text{integers modulo } n \text{ (i.e. } \mathbb{Z}/\mathbb{Z}_n); \\ \text{general linear group of dimension } n \text{ over a field } \Bbbk; \\ \text{special linear group of dimension } n \text{ over a field } \Bbbk; \\ \text{projective linear group of dimension } n \text{ over a field } \Bbbk; \\ \text{projective special linear group of dimension } n \text{ over a field } \Bbbk; \\ \text{group of all regular self-maps on a variety } X; \\ \text{group of all rational self-maps on a variety } X; \\ \text{Cremona group of rank } n \text{ over a field } \Bbbk \ \big(\text{identical to } \mathrm{Bir}(\mathbb{P}^{\mathrm{n}}_{\Bbbk})\big); \\ \text{group of all Cartier divisor classes on } X \text{ modulo linear equivalence.} \end{array}\right.$$

Chapter 6

Exceptional del Pezzo Surfaces

In this Chapter we explore the G-exceptionality of smooth del Pezzo G-surfaces by calculating their global G-invariant log canonical thresholds. These calculations involve examining explicitly the group structure and equations of such surfaces, providing proof of the results in Section 3.4. Tables of the global G-invariant log canonical thresholds calculated here can be found in Chapter 7.

6.1 Degree One

6.1.1 Background

Let X be a smooth del Pezzo surface of degree 1. Then X is a degree six hyper-surface in the weighted projective space $\mathbb{P}(1,1,2,3)$ with homogeneous co-ordinates x, y, z and t of weights $1,1,2,3$ respectively (Proposition 77). The bi-anti-canonical linear system is base point free and its corresponding map gives X as a double cover of a quadratic cone, $\mathcal{Q}$, in $\mathbb{P}^3$. By completing the square and the cube, we see that any such surface may be given by an equation of the form

$$t^2 + z^3 + zf_4(x,y) + f_6(x,y) = 0$$

where $f_k(x,y)$ is a homogeneous polynomial of degree k in the variables x and y.

Let G be the group of automorphisms Aut(X) of our surface X, then G is finite by Lemma 78 and always contains the subgroup $\mathbb{Z}_2$ generated by the Bertini involution β, that swaps the sheets of the double cover of the quadratic cone (see Section 5.1.4). Details of the possible automorphism groups realising minimal pairs (X, G) and their corresponding equations and generators can be found in [DI10].

Remarks 99.

- Observe that if $f_6(x, y) = 0$, then the surface X will be singular. Indeed, if $f_6(x, y) = 0$ then the ramification divisor of the double cover of $\mathbb{P}(1, 1, 2)$ is given by $z\big(z^2 + f_4(x, y)\big) = 0$.
- By similar reasoning; if a common root of f_4 and f_6 is a multiple root of f_6, then X will be singular.

6.1.2 General results

We answer here completely our Questions A and C by calculating the global log canonical thresholds of the G-surfaces (X, G) as G runs through all possible minimal automorphism groups.

Lemma 100. *The pluri-anti-canonical linear system* $|-2K_X|$ *contains* G*-invariant members, that is,* $|-2K_X|^G \neq \emptyset$.

Proof. As we remarked above, a del Pezzo surface of degree one is given by the zeros of $t^2 + z^3 + zf_4(x, y) + f_6(x, y)$ — we claim this form is canonical (in the sense that if we choose to write it without a z^2 term then the equation is unique). Indeed, if we are to make any non-trivial change of coordinates sending z to $\xi z + f_2(x, y)$ for $\xi \in \mathbb{C}$. Then we introduce a z^2 term back into the equation — completing the square and eliminating this term brings us back to our original equation. From this canonicity, it follows that for any automorphism $g \in G$, $g(z) = \nu z$ for some $\nu \in \mathbb{C}$. Observe then that the divisor $D = \{z = 0\}|_X = \{t^2 + f_6(x, y) = 0\}$ is a G-invariant member of $|-2K_X|$. □

Corollary 101. $\mathrm{lct}\big(X, G\big) \leqslant 2$ *for all possible automorphism groups* G.

From Cheltsov (Theorem 33) we know that

$$\mathrm{lct}(X,I) = \begin{cases} \frac{5}{6} & \text{if } |-K_X| \text{ contains cuspidal curves} \\ 1 & \text{if } |-K_X| \text{ contains no cuspidal curves} \end{cases}$$

where I is the trivial group.

It follows from this and Corollary 101 above that

$$\frac{5}{6} \leqslant \mathrm{lct}(X,G) \leqslant 2. \tag{6.1}$$

Lemma 102. *Suppose that there exists* $C \in |-K_X|^G$. *Then* $\mathrm{lct}(X,G) = \mathrm{lct}_1(X,G)$.

Proof. Observe first that if $\mathrm{lct}_1(X,G) = \frac{5}{6}$, then $\mathrm{lct}(X,G) = \mathrm{lct}_1(X,G)$ immediately by Theorem 33. We may assume then that $\mathrm{lct}_1(X,G) > \frac{5}{6}$, that is $\mathrm{lct}_1(X,G) = 1$.

Suppose that there exists $\lambda \in \mathbb{Q}$ such that $\mathrm{lct}(X,G) < \lambda < \mathrm{lct}_1(X,G) = 1$. Then there exists a G-invariant effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X, \lambda D)$ is not log canonical.

By Lemma 96, $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})$ is zero-dimensional. Set $H = (\lambda - 1)K_X$. Then $K_X + \lambda D + H \sim_{\mathbb{Q}} L = \mathcal{O}_X$ is Cartier and H is nef and big and we may apply Corollary 93. Whence $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})$ consists of at most $h^0(X, \mathcal{O}_X) = 1$ point, P.

By Corollary 94, P lies on a curve in the anti-canonical linear system. There are two possibilities for this point; either P is the base locus $\mathrm{Bs}(|-\mathrm{K_X}|)$ of the anti-canonical linear system $|-K_X|$, or it lies on an element of $|-K_X|$ outside of $\mathrm{Bs}(|-\mathrm{K_X}|)$. The point P cannot lie in the base locus, since this would contradict Corollary 94. If it lies on some element $\Omega \in |-K_X|$ outside this locus, then as P maps to itself under the group action, $\Omega \in |-K_X|^G$. We may assume that $\Omega \not\subseteq \mathrm{Supp}(D)$ by Convexity (Lemma 5). However,

$$1 > \lambda = \lambda D \cdot \Omega \geqslant \lambda \mathrm{mult}_P D > 1. \qquad \square$$

Lemma 103. *Suppose that* $|-K_X|^G = \emptyset$. *Then* $\mathrm{lct}(X,G) = \mathrm{lct}_2(X,G)$.

Proof. By Lemma 101, $|-2K_X|^G$ is non-empty. Suppose that there exists $\lambda \in \mathbb{Q}$ such that

$\mathrm{lct}(X,G) < \lambda < \mathrm{lct}_2(X,G) \leqslant 2$. Then there exists a G-invariant effective $\mathbb{Q}$-divisor

$$D = \sum_{i=0}^{r} d_i D_i \equiv -K_X,$$

where $d_i \in \mathbb{Q}_+$ and D_i are prime Weil divisors, such that the pair $(X, \lambda D)$ is not log canonical. There are two possibilities for the pair to fail to achieve log canonicity; either some component D_k of D has large coefficient, or D has a point of high multiplicity.

By Lemma 96, $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})$ is zero-dimensional and by Corollary 93 consists of at most two points.

Suppose $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})$ consists of exactly one point. It cannot be the base point of $|-K_X|$, as this contradicts Corollary 94. However, if the point lies on some element of $|-K_X|$ and is not the base locus, then as it maps to itself under the group action, this element in $|-K_X|$ must be G-invariant — contradicting the fact that $|-K_X|^G$ is empty.

Hence $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})$ consists of two points, P_1 and P_2 say. By similar arguments, neither P_1 nor P_2 can be the base point of $|-K_X|$ and there is no element $C \in |-K_X|$ such that $P_1, P_2 \in C$.

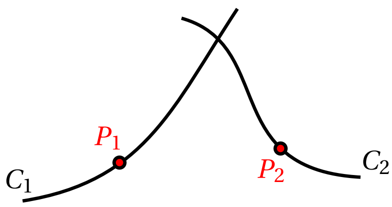


Figure 6.1: The two points of $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})$.

Thus, there are distinct curves C_1, C_2 elements of $|-K_X|$ such that $P_1 \in C_1$, $P_2 \in C_2$, $P_1, P_2 \notin \mathrm{Bs}(|-\mathrm{K_X}|)$ and $C_1 + C_2$ is a G-orbit of C_1 — this is shown in Figure 6.1. By Convexity (Lemma 5), we may assume that $C_1, C_2 \nsubseteq \mathrm{Supp}(D)$. Indeed, suppose that $C_1, C_2 \subseteq \mathrm{Supp}(D)$. Since the group action maps C_1 to C_2 we see that $C_1 + C_2 \in |-2K_X|^G$. Hence $(X, \lambda(C_1 + C_2))$ is log canonical. Writing $D \equiv \epsilon(C_1 + C_2) + \Delta$ (C_1 and C_2 have the same coefficient in D as they form a G-orbit) we see that $\Delta \equiv -(1-2\epsilon)K_X$. Intersecting D with $-K_X$ we see that $\frac{1}{2} > \epsilon$ ($\geqslant 0$). The pair $(X, \frac{\lambda}{(1-2\epsilon)}\Delta)$ is log canonical since the points of $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D}) \nsubseteq \mathrm{Supp}(\Delta)$. However, now $(X, \lambda D) = (X, \lambda(\epsilon(C_1 + C_2) + \Delta))$ is a weighted sum of log canonical pairs. This contradicts Convexity (Lemma 5). In fact, the same argument shows we may assume that

$\mathrm{Supp}(D)$ doesn't contain any curve in $|-2K_X|^G$.

For $i = 1, 2$, intersecting C_i with D we see that

$$1 = C_i \cdot D \geqslant \mathrm{mult}_{P_i} D \mathrm{mult}_{P_i} C_i \geqslant \mathrm{mult}_{P_i} D,$$

that is $\mathrm{mult}_{P_i} D \leqslant 1$ (of course, we also have that $\mathrm{mult}_{P_i} D > \frac{1}{\lambda}$).

Let $\sigma : \overline{X} \longrightarrow X$ be the blow-up of X at the points P_1, P_2 such that for $i = 1, 2$ $\sigma^*(P_i) = E_i$ are the exceptional divisors. From the equivalence

$$\sigma^*\big(K_X + \lambda D\big) \equiv K_{\overline{X}} + \lambda \overline{D} + \big(\lambda \mathrm{mult}_{P_1} D - 1\big) E_1 + \big(\lambda \mathrm{mult}_{P_2} D - 1\big) E_2$$

we see that the pair

$$\Big(\overline{X}, \lambda \overline{D} + \big(\lambda \mathrm{mult}_{P_1} D - 1\big) E_1 + \big(\lambda \mathrm{mult}_{P_2} D - 1\big) E_2\Big)$$

is not log canonical at finitely many points $Q_1, \ldots, Q_j$ for some $j \in \mathbb{N}$. If for $i = 1, 2$ and $1 \leqslant k \leqslant j$, $Q_k \in \overline{C_i}$ then by intersecting $\overline{D}$ with $\overline{C_i}$ and using Remark 11 we obtain

$$1 < \frac{2}{\lambda} < \mathrm{mult}_{Q_k} \overline{D} + \mathrm{mult}_{P_i} D \leqslant 1$$

— a contradiction. Hence $Q_k \notin \overline{C_i}$.

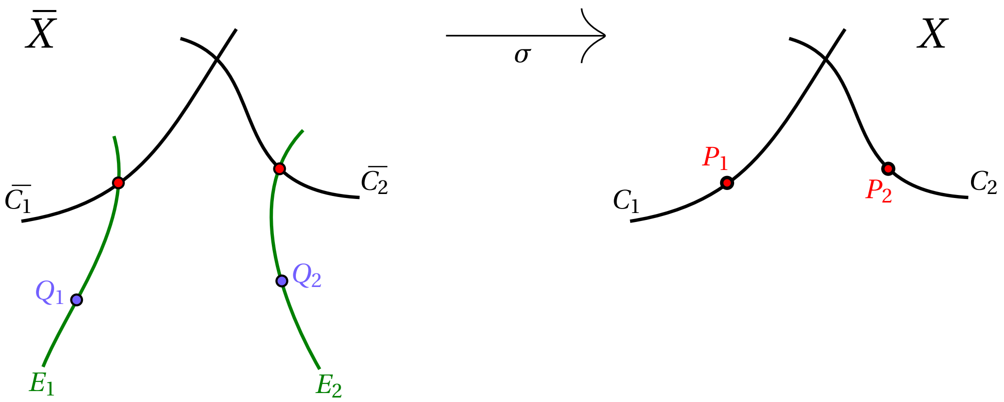


Figure 6.2: The blow-up of X at the two points of $\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})$.

It follows from Lemmata 81 and 82, that $j = 2$ and $Q_1 \in E_1, Q_2 \in E_2$, as depicted in

Figure 6.2. Since the points Q_1, Q_2 are unique on E_1, E_2, respectively then they must be invariant under the natural $\mathbb{Z}_2$-action on X (Bertini involution) and thus belong to the strict transform of the ramification divisor $\bar{R} \in \bar{X}$, this is a smooth curve on $\bar{X}$.

Our next observation is that, by Convexity (Lemma 5), we may assume that $\bar{R} \not\subseteq \mathrm{Supp}(\bar{D})$. Considering the intersection of $\bar{D}$ with $\bar{R}$ yields

$$\mathrm{mult}_{Q_1}\bar{D} + \mathrm{mult}_{Q_2}\bar{D} \leqslant \bar{D}\cdot\bar{R} = 3 - \mathrm{mult}_{P_1}D - \mathrm{mult}_{P_2}D$$

and as $P_1 + P_2$, $Q_1 + Q_2$ are two G-orbits we see that $\mathrm{mult}_{P_1}D = \mathrm{mult}_{P_2}D$ and $\mathrm{mult}_{Q_1}\bar{D} = \mathrm{mult}_{Q_2}\bar{D}$. Hence,

$$\mathrm{mult}_{Q_1}\bar{D} + \mathrm{mult}_{P_1}D \leqslant \frac{3}{2}$$

By Remark 11, we also have that $\mathrm{mult}_{Q_1}\bar{D} + \mathrm{mult}_{P_1}D > \frac{2}{\lambda}$. Thus,

$$\frac{3}{2} \geqslant \mathrm{mult}_{Q_1}\bar{D} + \mathrm{mult}_{P_1}D > \frac{2}{\lambda} \tag{6.2}$$

Let $\psi : X \xrightarrow{2:1} \mathcal{Q}$ be the map given by the linear system $|-2K_X|$. That is, a double cover of X over a quadric cone $\mathcal{Q} \cong \mathbb{P}(1,1,2) \subseteq \mathbb{P}^3$ ramified in R, the pre-image of a complete intersection of $\mathcal{Q}$ with a cubic in $\mathbb{P}^3$. We construct a unique hyperplane section of $\mathcal{Q}$ in the following way; take the line tangent to $\psi(R)$ at $\psi(P_1)$ and form the plane (shown in Figure 6.3) through this line and the point $\psi(P_2)$ — call the intersection curve of this plane with $\mathcal{Q}$ the conic $\psi(Z_1)$. In a similar way we may produce a curve $\psi(Z_2)$.

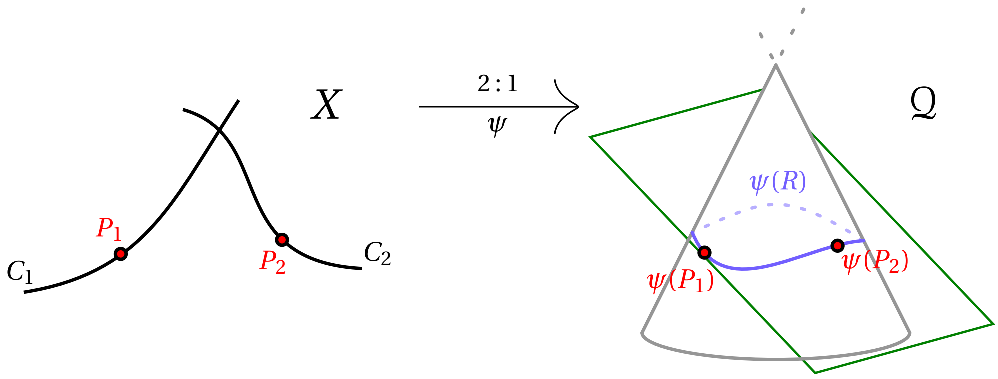


Figure 6.3: Construction of the curve $Z_1 \subseteq X$.

For $i = 1,2$, we consider the possibilities for these G-invariant curves upstairs on X:

- $Z_i = C_1 + C_2 \in |-2K_X|^G$, i.e. $\psi(Z_i)$ are lines on $\mathcal{Q}$;
- $Z_1 = Z_2$, $Z_i \in |-2K_X|^G$;
- $Z_1 \neq Z_2$, $Z_1 + Z_2 \in |-4K_X|^G$.

The first case can only occur when the C_i are tangent to R at the P_i. That is to say the C_i are singular at the points P_i (for $i = 1,2$), as ψ is a double cover of $\mathcal{Q}$. We may exclude this case by observing that as $C_i \nsubseteq \mathrm{Supp}(D)$,

$$2 = \big(C_1 + C_2\big) \cdot D \geqslant \sum_i \mathrm{mult}_{P_i} C_i \mathrm{mult}_{P_i} D > \frac{4}{\lambda} > 2.$$

In the second case, where $Z_1 = Z_2 \in |-2K_X|$, we note that Z_1 is G-irreducible[1] (since $\psi(Z_1)$ is a conic that is not a sum of 2 lines). Thus we may assume that $Z_1 \nsubseteq \mathrm{Supp}(D)$ by Convexity (Lemma 5). Since $\mathrm{mult}_{P_1} Z_1 = \mathrm{mult}_{P_2} Z_1 = 2$ and $\lambda < \mathrm{lct}_2\big(X, G\big)$ we see that

$$2 = Z_1 \cdot D \geqslant 2\mathrm{mult}_{P_1} D + 2\mathrm{mult}_{P_2} D = 4\mathrm{mult}_{P_1} D > \frac{4}{\lambda} > 2$$

— a contradiction.

Thus, the only possible case is for $Z_1 \neq Z_2$, $Z_1 + Z_2 \in |-4K_X|$.

Suppose that $\Big(X, \frac{\lambda}{4}\big(Z_1 + Z_2\big)\Big)$ is log canonical. Before we noted that $\mathrm{mult}_{P_i} D \leqslant 1$. Observe that $Z_1 + Z_2$ is G-irreducible since $Z_1 + Z_2$ is a G-orbit on X, thus by Convexity we may refine this inequality by intersecting $\overline{D}$ with $\overline{Z_1} + \overline{Z_2}$.

$$\begin{aligned} 0 \leqslant \overline{D} \cdot \big(\overline{Z_1} + \overline{Z_2}\big) &= 4 - \big(\mathrm{mult}_{P_1} D\big)\big(\mathrm{mult}_{P_1}(Z_1 + Z_2)\big) - \big(\mathrm{mult}_{P_2} D\big)\big(\mathrm{mult}_{P_2}(Z_1 + Z_2)\big) \\ &= 4 - 3\big(\mathrm{mult}_{P_1} D + \mathrm{mult}_{P_2} D\big) \end{aligned}$$

that is

$$\mathrm{mult}_{P_1} D = \mathrm{mult}_{P_2} D \leqslant \frac{2}{3}. \tag{6.3}$$

To derive our contradiction we'll consider the blow-up of $\overline{X}$ at the points Q_i. Let $\pi : \widetilde{X} \longrightarrow$

[1] A G-invariant curve C on a G-variety V is G-*irreducible* if $C = \sum_{i=0}^{r} C_i$ where the C_i are irreducible and belong to a single G-orbit.

$\overline{X}$ be this blow-up, with exceptional divisors F_i over Q_i — as in Figure 6.4. We may apply the same arguments to $\widetilde{X}$ as we did to $\overline{X}$, drawing the following conclusions:

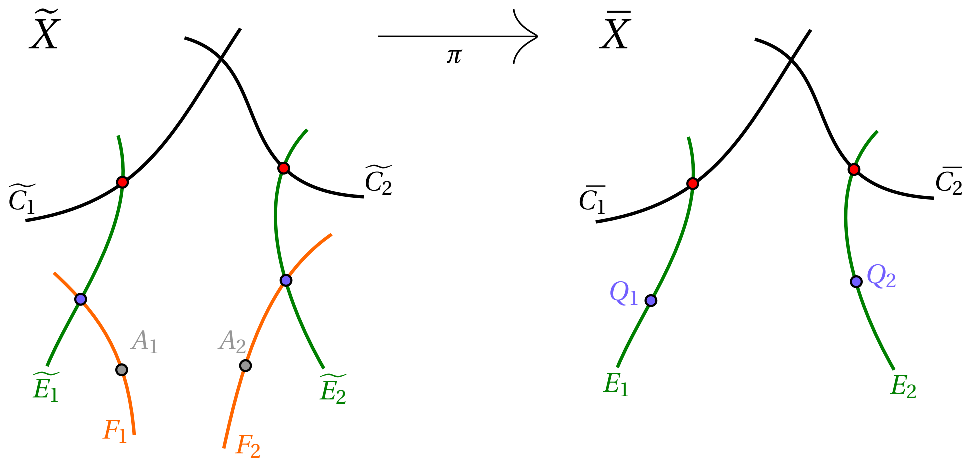


Figure 6.4: The blow-up of the points Q_1, Q_2 on $\overline{X}$.

- By considering the log pull-back under π

$$\pi^*\Big(K_{\overline{X}} + \lambda\overline{D} + \big(\lambda\mathrm{mult}_{P_1}D - 1\big)E_1 + \big(\lambda\mathrm{mult}_{P_2}D - 1\big)E_2\Big)$$
$$\equiv$$
$$K_{\widetilde{X}} + \lambda\widetilde{D} + \textstyle\sum_{i=1}^{2}\big(\lambda\mathrm{mult}_{P_i}D - 1\big)\widetilde{E}_i + \sum_{i=1}^{2}\big(\lambda\mathrm{mult}_{Q_i}\overline{D} + \lambda\mathrm{mult}_{P_i}D - 2\big)F_i$$

  we see that the pair

$$\left(\widetilde{X}, \lambda\widetilde{D} + \sum_{i=1}^{2}(\lambda\mathrm{mult}_{P_i}D - 1)\widetilde{E}_i + \sum_{i=1}^{2}(\lambda\mathrm{mult}_{Q_i}\overline{D} + \lambda\mathrm{mult}_{P_i}D - 2)F_i\right) = \big(\widetilde{X}, \Xi\big)$$

  is not log canonical at a finite number of points $A_1, \ldots, A_q$ since for $i = 1, 2$,

$$\left.\begin{aligned}\lambda\mathrm{mult}_{Q_i}\overline{D} + \lambda\mathrm{mult}_{P_i}D - 2\\ \lambda\mathrm{mult}_{P_i}D - 1\end{aligned}\right\} \leqslant 1.$$

  Moreover, by Lemmata 81 and 82, there is exactly one point A_1, A_2 on each of F_1, F_2, respectively and the natural $\mathbb{Z}_2$-action (Bertini involution) lifted on $\widetilde{X}$ has two fixed points on each F_i — the points A_i and $\widetilde{Q_i}$ for $i = 1, 2$.

- By using adjunction, we see that any G-orbit of the point A_1 (or A_2) consists of at most these two points.

Let $i = 1,2$, as before there are two possibilities for the A_i; either $A_i = \widetilde{E}_i \cap F_i$, or $A_i \in \widetilde{R}$. Suppose that $A_i = \widetilde{E}_i \cap F_i$.

Then as the pair $\left(\widetilde{X}, \Xi\right)$ is not log canonical at the point A_i, by adjunction neither is the pair

$$\left(\widetilde{E}_i, \lambda\widetilde{D}|_{\widetilde{E}_i} + (\lambda \mathrm{mult}_{Q_i}\bar{D} + \lambda \mathrm{mult}_{P_i} D - 2)F_i|_{\widetilde{E}_i}\right).$$

Hence

$$\left(\lambda\widetilde{D} + (\lambda \mathrm{mult}_{Q_i}\bar{D} + \lambda \mathrm{mult}_{P_i} D - 2)F_i\right) \cdot \widetilde{E}_i > 1$$

which implies that

$$\mathrm{mult}_{P_i} D > \frac{3}{4}$$

since $\widetilde{D} \cdot \widetilde{E_i} = \mathrm{mult}_{P_i} D - \mathrm{mult}_{Q_i}\bar{D}$ — contradicting inequality (6.3): $\mathrm{mult}_{P_i} D \leqslant \frac{2}{3}$.

Now $A_1, A_2 \in \widetilde{R}$ and since the pair $\left(\widetilde{X}, \Xi\right)$ is not log canonical at the points A_1, A_2 we have may re-write inequality (6.2) as

$$\mathrm{mult}_{A_i}\widetilde{D} + \mathrm{mult}_{P_i} D + \mathrm{mult}_{Q_i}\bar{D} - \frac{2}{\lambda} > \frac{1}{\lambda},$$

that is

$$\mathrm{mult}_{A_i}\widetilde{D} > \frac{3}{\lambda} - \mathrm{mult}_{P_i} D - \mathrm{mult}_{Q_i}\bar{D}. \tag{6.4}$$

Intersecting $\widetilde{R}$ with $\widetilde{D}$ we find a contradiction

$$\begin{aligned} 3 - (\mathrm{mult}_{P_1} D + \mathrm{mult}_{P_2} D + \mathrm{mult}_{Q_1}\bar{D} + \mathrm{mult}_{Q_2}\bar{D}) &= \widetilde{R} \cdot \widetilde{D} \\ &\geqslant \mathrm{mult}_{A_1}\widetilde{D} + \mathrm{mult}_{A_2}\widetilde{D} \\ &> 2(\frac{3}{\lambda} - \mathrm{mult}_{P_1} D - \mathrm{mult}_{Q_1}\bar{D}) \end{aligned}$$

and since $\lambda < 2$ we have

$$3 > \frac{6}{\lambda} > 3.$$

Thus our only possible escape from contradiction is for the pair $\left(X, \frac{\lambda}{4}\left(Z_1 + Z_2\right)\right)$ to not be log canonical. Suppose then that $\left(X, \frac{\lambda}{4}\left(Z_1 + Z_2\right)\right)$ is not log canonical.

We may run our arguments for the divisor D again for $D_S = \frac{1}{4}(Z_1 + Z_2)$. We encounter no problems up to, and including, equation (6.2). Namely, we have that $\mathrm{mult}_{P_1} D_S = \mathrm{mult}_{P_2} D_S$; $\mathrm{mult}_{Q_1} \overline{D_S} = \mathrm{mult}_{Q_2} \overline{D_S}$ and

$$\frac{3}{2} \geqslant \mathrm{mult}_{Q_1} \overline{D_S} + \mathrm{mult}_{P_1} D_S > \frac{2}{\lambda} > 1. \tag{6.5}$$

Since $\mathrm{mult}_{P_1}(Z_1 + Z_2) = 3$, $\mathrm{mult}_{P_1} D_S = \frac{3}{4}$ and together with equation (6.5) we find that

$$\frac{3}{4} \geqslant \mathrm{mult}_{Q_1} \overline{D_S} > \frac{1}{4}.$$

Hence $\mathrm{mult}_{Q_1} \overline{D_S} = \frac{2}{4}$ or $\frac{3}{4}$. However, $\mathrm{mult}_{Q_1}(\overline{Z_1} + \overline{Z_2}) = 0, 1$ or 2. Therefore, $\mathrm{mult}_{Q_1} \overline{D_S} = \frac{2}{4} = \frac{1}{2}$.

Now that we have obtained values for $\mathrm{mult}_{P_1} D_S$ and $\mathrm{mult}_{Q_1} \overline{D_S}$ replacing inequality (6.3), we continue by blowing up at the points Q_1, Q_2 as before. This leads to a contradiction on intersecting $\widetilde{R}$ with $\widetilde{D_S}$. Indeed,

$$\begin{aligned}
\frac{1}{2} &= 3 - \left(2\mathrm{mult}_{P_1} D_S + 2\mathrm{mult}_{Q_1} \overline{D_S}\right) \\
&= \widetilde{R} \cdot \widetilde{D_S} \\
&\geqslant \mathrm{mult}_{A_1} \widetilde{D_S} + \mathrm{mult}_{A_2} \widetilde{D_S} \\
&> \frac{2}{\lambda} \\
&> 1.
\end{aligned}$$

□

Theorem 104. *Let X be a general*[2] *smooth minimal del Pezzo G-surface of degree one with the prescribed automorphism group G, then*

$$\mathrm{lct}\big(X,\mathrm{Aut(X)}\big)=\begin{cases} 1 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_2,\\ 1 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_2\times\mathbb{Z}_2,\\ 1 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_4,\\ 1 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_6,\\ 1 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_4\times\mathbb{Z}_2,\\ \frac{5}{6} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_8,\\ \frac{5}{6} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_{10},\\ 1 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_2\times\mathbb{Z}_6,\\ \frac{5}{6} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_{12},\\ \frac{5}{6} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_{20},\\ 1 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_2\times\mathbb{Z}_{12},\\ \frac{5}{6} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_{24},\\ \frac{5}{6} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_{30},\\ 2 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{D}_8,\\ 2 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_2\bullet\mathbb{D}_4,\\ 2 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{D}_{12},\\ \frac{5}{3} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{D}_{16},\\ 2 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_2\bullet\mathbb{D}_{12},\\ 2 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_2\bullet\mathbb{A}_4,\\ \frac{5}{3} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_3\times\mathbb{D}_8,\\ 2 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_2\times\mathbb{Z}_3\bullet\mathbb{D}_6,\\ 2 & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_6\bullet\mathbb{D}_{12},\\ \frac{5}{3} & \textit{if}\,\mathrm{Aut(X)}=\mathbb{Z}_3\times\mathbb{Z}_2\bullet\mathbb{S}_4. \end{cases}$$

The following theorem lists the group-invariant global log canonical thresholds for special cases of smooth del Pezzo surfaces of degree one.

[2] The required generality is made explicit in restrictions on the parameters of the defining equations of X — that is 'general' means not on the list of Theorem 105.

Theorem 105. *Let X be a smooth minimal del Pezzo G-surface of degree one, then*

$$\operatorname{lct}\big(X,\mathrm{Aut(X)}\big)=\begin{cases}\frac{5}{6} & \text{if } \mathrm{Aut(X)}=\mathbb{Z}_2 \text{ and } |-K_X|^G \text{ contains cusps;}\\ \frac{5}{6} & \text{if } \mathrm{Aut(X)}=\mathbb{Z}_4 \text{ and } X \text{ can be described by the zeros of}\\ & t^2+z^3+z\big(ax^4+bx^2y^2+cy^4\big)+xy\big(dx^4+ex^2y^2+fy^4\big)\\ & \text{with either } a=0 \text{ or } c=0 \text{ (but not } a=c=0 \text{ and } d=f \text{ concurrently);}\\ \frac{5}{3} & \text{if } \mathrm{Aut(X)}=\mathbb{D}_8 \text{ and } X \text{ can be described by the zeros of}\\ & t^2+z^3+bzx^2y^2+xy\Big(c\big(x^4+y^4\big)+dx^2y^2\Big)\\ & \text{with } c\neq 0 \text{ and } d\neq 0;\\ \frac{5}{6} & \text{if } \mathrm{Aut(X)}=\mathbb{Z}_2{\bullet}\mathbb{D}_4 \text{ and } X \text{ can be described by the zeros of}\\ & t^2+z^3+z\Big(a\big(x^4+y^4\big)+bx^2y^2\Big)+xy\big(x^4-y^4\big)\\ & \text{with } a=0 \text{ or } 2(1+\epsilon_4^k)+\epsilon_4^{2k}b=0;\\ \frac{5}{3} & \text{if } \mathrm{Aut(X)}=\mathbb{Z}_2{\bullet}\mathbb{D}_4 \text{ and } X \text{ can be described by the zeros of}\\ & t^2+z^3+z\Big(a\big(x^4+y^4\big)+bx^2y^2\Big)+xy\big(x^4-y^4\big)\\ & \text{with } 2a\pm b=0.\end{cases}$$

Remark 106. It is not clear whether all the possiblities of Theorem 105 actually occur or not. For example, in the case when $G=\mathbb{D}_8$ (Section 6.1.3.14) the surface X is given by the zeros of $t^2+z^3+z\big(a(x^4+y^4)+bx^2y^2\big)+xy\big(c(x^4+y^4)+dx^2y^2\big)$. If $a=0$ then the surface can contain cuspidal curves in $|-2K_X|^G$ whilst being non-singular. It is not clear however if with this choice of a we allow more automorphisms and causing the size of Aut(X) = G to jump.

6.1.3 Results for individual automorphism groups

Let X be a smooth minimal del Pezzo G-surface of degree one such that $G=\mathrm{Aut(X)}$ and x,y,z,t be homogeneous coordinates on $\mathbb{P}(1,1,2,3)$ with weights $1,1,2,3$, respectively. Denote the automorphism $\varphi: X\longrightarrow X$ mapping

$$(x:y:z:t)\mapsto\big(\varphi(x):\varphi(y):\varphi(z):\varphi(t)\big)\quad\text{by}\quad\big[\varphi(x),\varphi(y),\varphi(z),\varphi(t)\big],$$

let $\epsilon_k=e^{\frac{2\pi i}{k}}$ be the k^{th} primitive root of unity and write β for the natural $\mathbb{Z}_2$-action (Bertini involution) $[x,y,z,-t]$. All notations are described in detail in Chapter 5.

6.1.3.1 Aut(X) = $\mathbb{Z}_2$

Lemma 107.

$$\mathrm{lct}\big(X,\mathbb{Z}_2\big)=\begin{cases}\frac{5}{6} & \textit{if } |-K_X| \textit{ contains cuspidal curves,}\\ 1 & \textit{if } |-K_X| \textit{ contains no cuspidal curves.}\end{cases}$$

Equation of surface and group action

Equation of X:

$$t^2+z^3+zf_4(x,y)+f_6(x,y)=0,$$

with $f_4(x,y)$ and $f_6(x,y)$ general such that Aut(X) is no bigger than $\mathbb{Z}_2$.

Generator of $\mathrm{Aut}(X)$ (Bertini involution):

$$\beta=[x,y,z,-t].$$

Proof of Lemma 107. As the automorphism group of any smooth del Pezzo surface of degree one contains $\mathbb{Z}_2$ as a subgroup (cf. Proposition 79), it follows that $\mathrm{lct}\big(X,\mathbb{Z}_2\big)=\mathrm{lct}\big(X,I\big)$, where I is the trivial group. □

6.1.3.2 Aut(X) = $\mathbb{Z}_2\times\mathbb{Z}_2$

Lemma 108.

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)=1.$$

Equation of surface and group action

Equation of X:

$$t^2+z^3+zf_4(x,y)+f_6(x,y)=0,$$

with $f_4(x,y)=ax^4+bx^2y^2+cy^4$ and $f_6(x,y)=dx^6+ex^4y^2+fx^2y^4+gy^6$.

Generators of $\mathrm{Aut}(X)$:

$$\beta, g=[x,-y,z,t].$$

Generality conditions Observe that if either $g = c = 0$, or $a = d = 0$ then, by Remark 99, we contradict the smoothness of X.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains the following two G-invariant curves

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2 + z^3 + cy^4z + gy^6 = 0\}, \\ C_2 &= \{y=0\}|_X = \{t^2 + z^3 + ax^4z + dx^6 = 0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big) = 1.$$

Proof. Observe that C_1 has singular points at $\left(0:1:\pm\sqrt{\frac{-c}{3}}:0\right)$, whenever $3g \pm 4c\sqrt{\frac{-c}{3}} = 0$. These are nodal unless $c = 0$, whence they are cuspidal. Similarly, C_2 has singular points $\left(1:0:\pm\sqrt{\frac{-a}{3}}:0\right)$ with $3d \pm 4a\sqrt{\frac{-a}{3}} = 0$ that are nodal unless $a = 0$, whence they are cuspidal. Neither of these cuspidal cases may occur due to the generality conditions above. □

6.1.3.3 Aut(X) = $\mathbb{Z}_4$

Lemma 109.

$$\mathrm{lct}\big(X,G\big) = \mathrm{lct}_1\big(X,G\big) = \begin{cases} \frac{5}{6} & \text{if} \quad a=0 \quad \text{and} \quad c \neq 0, \\ \frac{5}{6} & \text{if} \quad a=0 \quad \text{and} \quad d \neq f, \\ \frac{5}{6} & \text{if} \quad c=0 \quad \text{and} \quad a \neq 0, \\ \frac{5}{6} & \text{if} \quad c=0 \quad \text{and} \quad d \neq f, \\ 1 & \text{otherwise.} \end{cases}$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x,y) + f_6(x,y) = 0,$$

with $f_4(x,y) = ax^4 + bx^2y^2 + cy^4$ and $f_6(x,y) = xy(dx^4 + ex^2y^2 + fy^4)$.

Generator of $\mathrm{Aut}(X)$:

$$g = [x, -y, -z, it].$$

Generality conditions If $a = 0$, $c = 0$ and $d = f$ then we have an extra automorphism of X. Indeed, if $a = c = 0$ then f_4 has two double roots at $(0,1)$ and $(1,0)$. Any proper symmetry preserving this set must swap these roots, that is send $(x,y) \mapsto (y,x)$. However, this will not leave invariant the roots of f_6 unless $d = f$.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x = 0\}|_X = \{t^2 + z^3 + cy^4z = 0\}, \\ C_2 &= \{y = 0\}|_X = \{t^2 + z^3 + ax^4z = 0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = \begin{cases} \frac{5}{6} & \text{if} \quad a = 0 \quad \text{and} \quad c \neq 0, \\ \frac{5}{6} & \text{if} \quad a = 0 \quad \text{and} \quad d \neq f, \\ \frac{5}{6} & \text{if} \quad c = 0 \quad \text{and} \quad a \neq 0, \\ \frac{5}{6} & \text{if} \quad c = 0 \quad \text{and} \quad d \neq f, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. By inspection, $C_i \in |-K_X|^G$ are smooth curves for $a, c \neq 0$ and cuspidal otherwise. From the generality conditions, we require that $a = 0$, $c = 0$ and $d = f$ don't occur concurrently. □

6.1.3.4 $\mathrm{Aut(X)} = \mathbb{Z}_6$

Lemma 110.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x,y) + f_6(x,y) = 0.$$

There are three cases, as $\mathbb{Z}_6$ can belong to three different conjugacy classes:

(i) $f_4(x,y) = x(ax^3 + by^3)$ and $f_6(x,y) = cx^6 + dx^3y^3 + ey^6$.

Generator of $\mathrm{Aut}(X)$:

$$g = [x, \epsilon_3 y, z, -t].$$

Generality conditions Observe that if either $e = 0$, or $a = c = 0$ then (by Remark 99) we contradict the smoothness of X.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x = 0\}|_X = \{t^2 + z^3 + ey^6 = 0\}, \\ C_2 &= \{y = 0\}|_X = \{t^2 + z^3 + ax^4z + cx^6 = 0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = 1.$$

Proof. Both curves $C_i \in |-K_X|^G$ are smooth, taking into consideration the generality conditions above. □

(ii) $f_4(x,y) = x^2y^2$ and $f_6(x,y) = ax^6 + bx^3y^3 + cy^6$.

Generator of $\mathrm{Aut}(X)$:

$$g = [x, \epsilon_3 y, \epsilon_3 z, -t].$$

Generality conditions Observe that if either $a = 0$, or $c = 0$ then (by Remark 99) we contradict the smoothness of X.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3+cy^6=0\}, \\ C_2 &= \{y=0\}|_X = \{t^2+z^3+ax^6=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big)=1.$$

Proof. Both curves $C_i \in |-K_X|^G$ are smooth, taking into consideration the generality conditions above. □

(iii) $f_4(x,y)=0$ and $f_6(x,y)=ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6$ such that f_6 is without multiple roots.

Generator of $\mathrm{Aut}(X)$:

$$g=[x,y,\epsilon_3 z,-t].$$

Generality conditions Observe that for $g=0$ (resp. $a=0$), if $f=0$ (resp. $b=0$) then $f_6(x,y)$ has multiple roots. However, $f\neq 0$ (resp. $b\neq 0$) then the ramification divisor of the double cover of a quadratic cone by X is singular. Hence both g and a are non-zero.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the entire anti-canonical linear system is G-invariant, that is

$$|-K_X|^G=\{\lambda x+\mu y=0\}|_X.$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big)=1.$$

Proof. Invariant curves in $|-K_X|$ are given by

$$C_{(\lambda:\mu)} = \{\lambda x + \mu y\}|_X,$$

for $(\lambda:\mu) \in \mathbb{P}^1$. As the equation for X is symmetric in x and y, we need consider only the case where $\mu \neq 0; \mu = 1$. Thus the curves C_λ are given by the zero locus of equations

$$t^2 + z^3 + x^6(a + b\lambda + c\lambda^2 + d\lambda^3 + e\lambda^4 + f\lambda^5 + g\lambda^6).$$

We see that for $a \neq 0$, there are no values of λ for which the curve C_λ is singular. Hence for $a \neq 0$ (and $g \neq 0$ by taking $\lambda \neq 0$) the claim follows; we may make these assumptions by the generality conditions above. □

6.1.3.5 Aut(X) = $\mathbb{Z}_4 \times \mathbb{Z}_2$

Lemma 111.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x, y) + f_6(x, y) = 0,$$

with $f_4(x, y) = ax^4 + by^4$ and $f_6(x, y) = x^2(cx^4 + dy^4)$.

Generators of Aut(X):

$$g = [ix, y, -z, it], \beta.$$

Generality conditions Observe that either b must be non-zero, or a and c must be non-zero. Indeed suppose that either $b = 0$, or $a = c = 0$, then by Remark 99, we contradict the smoothness of X.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3+by^4z=0\},\\ C_2 &= \{y=0\}|_X = \{t^2+z^3+ax^4z+cx^6=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big)=1.$$

Proof. Notice that C_1 is cuspidal when $b=0$ and smooth otherwise. C_2 has singular points at $\left(1:0:\pm\sqrt{\frac{-a}{3}}:0\right)$ when $3c\pm 2a\sqrt{\frac{-a}{3}}=0$. At these points the curve is cuspidal when $a=0$ and nodal otherwise. □

6.1.3.6 Aut(X) = $\mathbb{Z}_8$

Lemma 112.

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)=\frac{5}{6}.$$

Equation of surface and group action

Equation of X:

$$t^2+z^3+zf_4(x,y)+f_6(x,y)=0,$$

with $f_4(x,y)=ax^2y^2$ and $f_6(x,y)=xy(cx^4+dy^4)$.

Generator of Aut(X):

$$g=[x,y,-iz,-\epsilon_8 t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the entire anti-canonical linear system is G-invariant, that is

$$|-K_X|^G=\{\lambda x+\mu y=0\}|_X.$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1(X,G) = \frac{5}{6}.$$

Proof. Invariant curves in $|-K_X|$ are given by

$$C_{(\lambda:\mu)} = \{\lambda x + \mu y\}|_X,$$

for $(\lambda:\mu) \in \mathbb{P}^1$. As the equation for X is symmetric in x and y, we may consider only the case where $\mu \neq 0; \mu = 1$. Thus the curves C_λ are given by the zero locus of equations

$$t^2 + z^3 + \lambda\big(\lambda a x^4 z + x^6(c + d\lambda^2)\big)$$

and taking $\lambda = 0$ we see that the anti-canonical linear system contains cusps. □

6.1.3.7 Aut(X) = $\mathbb{Z}_{10}$

Lemma 113.

$$\mathrm{lct}(X,G) = \mathrm{lct}_1(X,G) = \frac{5}{6}.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + z f_4(x,y) + f_6(x,y) = 0,$$

with $f_4(x,y) = ax^4$ and $f_6(x,y) = x(bx^5 + y^5)$.

Generator of Aut(X):

$$g = [x, \epsilon_5 y, z, -t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3=0\}, \\ C_2 &= \{y=0\}|_X = \{t^2+z^3+ax^4z+bx^6=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big) = \frac{5}{6}.$$

Proof. By inspection, the curve C_1 is cuspidal. □

6.1.3.8 Aut(X) = $\mathbb{Z}_2 \times \mathbb{Z}_6$

Lemma 114.

$$\mathrm{lct}\big(X,G\big) = \mathrm{lct}_1\big(X,G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$t^2+z^3+zf_4(x,y)+f_6(x,y)=0.$$

There are three cases, corresponding to three possible conjugacy classes:

(i) with $f_4(x,y)=x^4$ and $f_6(x,y)=ax^6+by^6$.

Generator of Aut(X):

$$g=[x,\epsilon_6 y,z,t].$$

Generality conditions Observe that b must be non-zero; indeed, if $b=0$ then (by Remark 99) this contradicts the smoothness of X.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3+by^6=0\}, \\ C_2 &= \{y=0\}|_X = \{t^2+z^3+x^4z+ax^6=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big)=1.$$

Proof. The curve C_1 is non-singular unless $b=0$, whence it is a cusp. C_2 is non-singular unless $3a \pm \frac{2i\sqrt{3}}{3}=0$, whence it is nodal. □

(ii) with $f_4(x,y)=x^2y^2$ and $f_6(x,y)=ax^6+by^6$.

Generators of $\mathrm{Aut}(X)$:

$$g=[\epsilon_6 x, y, \epsilon_3^2 z, t], \beta.$$

Generality conditions Observe that a and b must be non-zero; indeed, if $a=0$ or $b=0$ then (by Remark 99) this contradicts the smoothness of X.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3+by^6=0\}, \\ C_2 &= \{y=0\}|_X = \{t^2+z^3+ax^6=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big)=1.$$

Proof. Both curves $C_1, C_2 \in |-K_X|^G$ are smooth, taking into consideration the generality conditions. □

(iii) with $f_4(x,y)=0$ and $f_6(x,y)=dx^6+ex^4y^2+fx^2y^4+gy^6$.

Generator of $\mathrm{Aut}(X)$:

$$g=[-x,y,\epsilon_3 z,t].$$

Generality conditions Observe that if $g=0$ or $d=0$ then the ramification divisor of the double cover of the quadratic cone in $\mathbb{P}^3$ is singular, contradiction the smoothness of X.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X=\{t^2+z^3+gy^6=0\},\\ C_2 &= \{y=0\}|_X=\{t^2+z^3+dx^6=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big)=1.$$

Proof. Both curves $C_1,C_2\in|-K_X|^G$ are smooth, taking into consideration the generality conditions. □

6.1.3.9 Aut(X) = $\mathbb{Z}_{12}$

Lemma 115.

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)=\frac{5}{6}.$$

Equation of surface and group action

Equation of X:

$$t^2+z^3+zf_4(x,y)+f_6(x,y)=0,$$

with $f_4(x,y)=0$ and $f_6(x,y)=xy(x^4+ax^2y^2+by^4)$.

Generator of $\mathrm{Aut}(X)$:

$$g = [-x, y, \epsilon_6 z, it].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3=0\}, \\ C_2 &= \{y=0\}|_X = \{t^2+z^3=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = \frac{5}{6}.$$

Proof. By inspection, both C_1 and C_2 are cusps. □

6.1.3.10 Aut(X) = $\mathbb{Z}_{20}$

Lemma 116.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = \frac{5}{6}.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + z f_4(x, y) + f_6(x, y) = 0,$$

with $f_4(x, y) = ax^4$ and $f_6(x, y) = xy^5$.

Generator of $\mathrm{Aut}(X)$:

$$g = [x, \epsilon_{10} y, -z, it].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3=0\},\\ C_2 &= \{y=0\}|_X = \{t^2+z^3+ax^4z=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big)=\frac{5}{6}.$$

Proof. By inspection, the curve C_1 is a cusp. □

6.1.3.11 Aut(X) = $\mathbb{Z}_2\times\mathbb{Z}_{12}$

Lemma 117.

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)=1.$$

Equation of surface and group action

Equation of X:

$$t^2+z^3+zf_4(x,y)+f_6(x,y)=0,$$

with $f_4(x,y)=x^4$ and $f_6(x,y)=y^6$.

Generators of Aut(X):

$$g=[\epsilon_{12}x,y,\epsilon_3^2z,-t],\beta.$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3+y^6=0\},\\ C_2 &= \{y=0\}|_X = \{t^2+z^3+x^4z=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1(X,G) = 1.$$

Proof. Both the curves C_1 and C_2 are smooth members of $|-K_X|^G$. □

6.1.3.12 Aut(X) = $\mathbb{Z}_{24}$

Lemma 118.

$$\mathrm{lct}(X,G) = \mathrm{lct}_1(X,G) = \frac{5}{6}.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x,y) + f_6(x,y) = 0,$$

with $f_4(x,y) = 0$ and $f_6(x,y) = xy(x^4 + by^4)$.

Generator of Aut(X):

$$g = [ix, y, \epsilon_{12}z, \epsilon_8 t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2+z^3=0\}, \\ C_2 &= \{y=0\}|_X = \{t^2+z^3=0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1(X,G) = \frac{5}{6}.$$

Proof. By inspection, both the curves C_1 and C_2 are cuspidal. □

6.1.3.13 Aut(X) = $\mathbb{Z}_{30}$

Lemma 119.

$$\mathrm{lct}\big(X,G\big) = \mathrm{lct}_1\big(X,G\big) = \frac{5}{6}.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x,y) + f_6(x,y) = 0,$$

with $f_4(x,y) = 0$ and $f_6(x,y) = x(x^5 + y^5)$.

Generator of Aut(X):

$$g = [x, \epsilon_5 y, \epsilon_3 z, -t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains two G-invariant curves, namely

$$\begin{aligned} C_1 &= \{x=0\}|_X = \{t^2 + z^3 = 0\},\\ C_2 &= \{y=0\}|_X = \{t^2 + z^3 + x^6 = 0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big) = \frac{5}{6}.$$

Proof. By inspection, C_1 is cuspidal and C_2 is non-singular. □

6.1.3.14 Aut(X) = $\mathbb{D}_8$

Lemma 120.

$$\mathrm{lct}\big(X,G\big) = \mathrm{lct}_2\big(X,G\big) = \begin{cases} \frac{5}{3} & \text{if} \quad a=0 \quad (\text{and} \quad c,d \neq 0),\\ 2 & \text{otherwise.} \end{cases}$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x, y) + f_6(x, y) = 0,$$

with $f_4(x, y) = a(x^4 + y^4) + bx^2y^2$ and $f_6(x, y) = xy\big(c(x^4 + y^4) + dx^2y^2\big)$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [y, -x, z, it], g_2 = [y, x, z, t].$$

Generality conditions Observe that if $c = d = 0$ then $f_6(x, y) = 0$ and by Remark 99 we contradict the smoothness of X. Furthermore if $2a \pm b = 0$ and $2c \pm d = 0$, then $(x^2 + y^2)^2$ is a shared root of f_4 and f_6 that is a multiple root of f_6 — this contradicts the smoothness of X (cf. Remarks 99). When $a = 0$, if $c = 0$ then X will be singular by Remark 99 since a shared root of f_4 and f_6 will be a double root of f_6. Further, if $a = d = 0$ then we allow extra automorphisms and the group jumps to $\mathbb{D}_{16}$.

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. $\mathbb{D}_8$ acts on $|-K_X| = \{\lambda x + \mu y = 0\}|_X$ $((\lambda : \mu) \in \mathbb{P}^1)$ yielding a 2-dim representation of $\mathbb{D}_8$ on $\mathbb{C}^2$. The generators g_1 and g_2 are given,respectively, by the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in this representation. We see that A has Eigenvalues $\mp i$ and corresponding Eigenvectors $\begin{bmatrix} 1 & \pm i \end{bmatrix}^T$. However

$$B \begin{bmatrix} 1 \\ \pm i \end{bmatrix} = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$$

and so there is no common Eigenspace between the generators. Thus the representation is irreducible, and the claim follows. □

Action of G on $|-2K_X|$

Claim. *Let*

$$\begin{aligned}
C_1 &= \{xy=0\}|_X = \{t^2+z^3+ax^4z=0\},\\
C_2 &= \{z=0\}|_X = \{t^2+xy(c(x^4+y^4)+dx^2y^2)=0\},\\
C_3 &= \{x^2+y^2=0\}|_X = \{t^2+z^3+x^4z(2a-b)\pm x^6 i(2c-d)=0\},\\
C_4 &= \{x^2-y^2=0\}|_X = \{t^2+z^3+x^4z(2a+b)\pm x^6 i(2c+d)=0\}.
\end{aligned}$$

Then the G-invariant curves in $|-2K_X|$ are C_1, C_4 and members of the pencil $\{\lambda C_2+\mu C_3=0\}$ for $(\lambda:\mu)\in\mathbb{P}^1$.

Proof. Since for $\Delta\in|-2K_X|$, $\Delta=\{\lambda_0x^2+\lambda_1y^2+\lambda_2xy+\lambda_3z=0\}|_X$ for some $(\lambda_0:\lambda_1:\lambda_2:\lambda_3)\in\mathbb{P}^3$ we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates x^2, y^2, xy and z). The generators g_1 and g_2 correspond to matrices

$$\Gamma_1=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\quad\text{and}\quad\Gamma_2=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

respectively. We readily see that these split in to

$$\Gamma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus\begin{bmatrix}-1\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}\quad\text{and}\quad\Gamma_2=\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}.$$

However, the representation splits further as

$$\Gamma_1=\begin{bmatrix}1\end{bmatrix}\oplus\begin{bmatrix}-1\end{bmatrix}\oplus\begin{bmatrix}-1\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}\quad\text{and}\quad\Gamma_2=\begin{bmatrix}1\end{bmatrix}\oplus\begin{bmatrix}-1\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}.$$

Thus, the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^1$. This gives us four 1-dimensional irreducible subspaces corresponding to the G-invariant curves $C_1=\{xy=0\}|_X$, $C_2=\{z=0\}|_X$, $C_3=\{x^2+y^2=0\}|_X$ and $C_4=\{x^2-y^2=0\}|_X$.

Since the curves C_2 and C_3 are compatible under the action of the group G, linear combinations of them belong to the linear system $|-2K_X|$.

Observe that the final two curves, C_3 and C_4, can also be seen as coming from the following: From the 2-dim representation of $\mathbb{D}_8$ on $\mathbb{C}^2$ coming from the action of $\mathbb{D}_8$ on $|-K_X|$, we see that corresponding to the Eigenvectors of the matrix giving the g_1 action we have two curves, D_1 and D_2 in $|-K_X|$ that are $\mathbb{Z}_4$-invariant ($\mathbb{Z}_4 = \langle g_1 \rangle$). The action of $\mathbb{Z}_2$ interchanges these curves, whence $C_3 = D_1 + D_2 \in |-2K_X|^G$. Similarly, if we start with the $\mathbb{Z}_2$-invariant curves corresponding to the Eigenvectors of the matrix representation of g_2 and then apply the $\mathbb{Z}_4$ action, we find that $C_4 = D_3 + D_4$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2\big(X,G\big) = \begin{cases} \frac{5}{3} & \text{if} \quad a = 0, \\ \frac{5}{3} & \text{if} \quad c = 2a - b = 0 \quad \text{and} \quad 4ia\sqrt{a}(3+(-1)^k) + (-1)^l 3\sqrt{3}d \neq 0, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. First we examine the curves C_1 and C_4 and determine how bad their singularities can be. Then we examine the curves in the pencil $\{\lambda C_2 + \mu C_3 = 0\}$, for $(\lambda : \mu) \in \mathbb{P}^1$.

(i) $C_1 = \{xy = 0\}|_X$.

Since the equation is symmetric in x and y we may assume without loss of generality that $y = 0$. Then C_1 is given by the zero locus of $t^2 + z^3 + ax^4z$. Clearly if $a = 0$ then C_1 is cuspidal, otherwise C_1 is a non-singular curve.

(ii) $C_4 = \{x^2 - y^2 = 0\}|_X = \{t^2 + z^3 + x^4z(2a+b) \pm x^6(2c+d) = 0\}$.

In a similar way, we calculate for C_4 that whenever $\pm 4(2a+b)\sqrt{\frac{-(2a+b)}{3}} \pm 6(2c+d) = 0$ the curve has singular points at $(1 : \pm 1 : \pm\sqrt{\frac{-(2a+b)}{3}} : 0)$ which are nodal unless $2a+b = 0$ — whence these points are cusps.

(iii) $\{\lambda C_2 + \mu C_3 = 0\}|_X = \{\lambda z + \mu(x^2 + y^2) = 0\}|_X$.

Suppose that $\lambda \neq 0; \lambda = -1$, then for a curve F in this pencil we may write

$$F = \Big\{ t^2 + x^6\big(\mu(\mu^2+a)\big) + cx^5y + x^4y^2\big(\mu(3\mu^2+a+b)\big) + dx^3y^3 \\ + x^2y^4\big(\mu(3\mu^2+a+b)\big) + cxy^5 + y^6\big(\mu(\mu^2+a)\big) = 0 \Big\}.$$

For F to have singularities worse than nodes, we must have that

$$\left.\begin{array}{ll} \mu(\mu^2+a) & = 0 \\ c & = 0 \\ \mu(3\mu^2+a+b) & = 0 \end{array}\right\}$$

that is either

$$c = 2a - b = 0,$$

or

$$c = \mu = 0.$$

The case $c = \mu = 0$ is a special case of the curve C_2, which we examine below. When $c = 2a - b = 0$, the curve F is given by the zeros of the polynomial $t^2 + dx^3y^3$. For

$$4ia\sqrt{a}(3 + (-1)^k) + (-1)^l 3\sqrt{3}d = 0,$$

the surface will be singular — a contradiction.

Thus F has cuspidal points when $c = 2a - b = 0$ and $4ia\sqrt{a}(3 + (-1)^k) + (-1)^l 3\sqrt{3}d \neq 0$. The only curves remaining to check are C_2 and C_3.

(iv) $C_3 = \{x^2 + y^2 = 0\}|_X = \{t^2 + z^3 + x^4z(2a-b) \pm x^6 i(2c-d) = 0\}$.

Here we calculate that the curve has singular points at $(1 : \pm i : \pm\sqrt{\frac{-(2a-b)}{3}} : 0)$ whenever $\pm 4(2a-b)\sqrt{\frac{-(2a-b)}{3}} \pm 6i(2c-d) = 0$ which are nodal if $2a - b \neq 0$ and cuspidal otherwise.

(v) $C_2 = \{z = 0\}|_X = \{t^2 + xy(c(x^4 + y^4) + dx^2y^2) = 0\}$.

For $2c \pm d \neq 0$, C_2 is a non-singular curve. Otherwise, we have singular points at $(1 : \epsilon_4^k : 0 : 0)$ which are nodal unless $c = d = 0$, whence they are cusps. □

6.1.3.15 $\mathrm{Aut(X)} = \mathbb{Z}_2 \bullet \mathbb{D}_4$

Lemma 121.

$$\mathrm{lct}\big(X,G\big) = \begin{cases} \frac{5}{6} & \text{if} \quad a = 0, \\ \frac{5}{6} & \text{if} \quad 2(1+\epsilon_4^k)+\epsilon_4^{2k} b = 0, \\ \frac{5}{3} & \text{if} \quad 2a \pm b = 0, \\ 2 & \text{otherwise.} \end{cases}$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + z f_4(x,y) + f_6(x,y) = 0,$$

with $f_4(x,y) = a(x^4+y^4) + bx^2y^2$ and $f_6(x,y) = xy(x^4-y^4)$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [x,-y,-z,it], g_2 = [y,x,-z,it].$$

Generality conditions Each of the conditions $a = 0$, $2(1+\epsilon_4^k)+\epsilon_4^{2k}b = 0$ and $2a \pm b = 0$ are possible without enlarging the size of the group, that is allowing extra automorphisms. In the case where $a = 0$, the equation is isomorphic to that of a del Pezzo of degree one with automorphism group $\mathbb{D}_{16}$ however this isomorphism is not $\mathbb{Z}_2 \bullet \mathbb{D}_4$-equivariant.

Action of G on $|-K_X|$

Claim. *$|-K_X|^G = \emptyset$, unless either $a = 0$ or $2(1+\epsilon_4^k)+\epsilon_4^{2k}b = 0$ (whence there is a cusp in the G-invariant anti-canonical linear system).*

Proof. First observe that if either $a = 0$ or $2(1+\epsilon_4^k)+\epsilon_4^{2k}b = 0$, then there is a cuspidal curve in the anti-canonical linear system.

G acts on $|-K_X| = \{\lambda x + \mu y = 0\}|_X$ for $(\lambda:\mu) \in \mathbb{P}^1$, yielding a 2-dim representation of G on

$\mathbb{C}^2$. The generators g_1 and g_2 are given,respectively, by the matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in this representation. We see that Γ_2 exchanges the Eigenvectors of Γ_1. Thus there is no common Eigenspace between the generators. Therefore the representation is irreducible, and the claim follows. $\square$

Action of G on $|-2K_X|$

Claim. *The following four curves are the only G-invariant members of $|-2K_X|$.*

$$\begin{aligned} C_1 &= \{xy = 0\}|_X = \{t^2 + z^3 + ax^4z = 0\}, \\ C_2 &= \{z = 0\}|_X = \{t^2 + xy(x^4 - y^4) = 0\}, \\ C_3 &= \{x^2 + y^2 = 0\}|_X = \{t^2 + z^3 + x^4z(2a - b) = 0\}, \\ C_4 &= \{x^2 - y^2 = 0\}|_X = \{t^2 + z^3 + x^4z(2a + b) = 0\}. \end{aligned}$$

Proof. Since for $\Delta \in |-2K_X|$, $\Delta = \{\lambda_0 x^2 + \lambda_1 y^2 + \lambda_2 xy + \lambda_3 z = 0\}|_X$ for some $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$ we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates x^2, y^2, xy and z). The generators g_1 and g_2 correspond to matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

respectively. We readily see that these split in to

$$\Gamma_1 = \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} -1 \end{bmatrix} \oplus \begin{bmatrix} -1 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} -1 \end{bmatrix}.$$

Thus, the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$. This gives us four 1-dimensional irreducible subspaces corresponding to the G-invariant curves $C_1 = \{xy = 0\}|_X$, $C_2 = \{cz = 0\}|_X$, $C_3 = \{x^2 + y^2 = 0\}|_X$ and $C_4 = \{x^2 - y^2 = 0\}|_X$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\operatorname{lct}_2(X,G) = \begin{cases} \frac{5}{3} & \text{if} \quad a = 0, \\ \frac{5}{3} & \text{if} \quad 2a \pm b = 0, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. We examine each of the curves C_1, C_2, C_3, C_4 in turn and determine how bad their singularities can be.

(i) $C_1 = \{xy = 0\}|_X$.

We may assume without loss of generality that $y = 0$. Then C_1 is given by the zero locus of $t^2 + z^3 + ax^4z$. Clearly if $a = 0$ then C_1 is cuspidal, otherwise C_1 is a non-singular curve.

(ii) $C_2 = \{z = 0\}|_X = \{t^2 + xy(x^4 - y^4) = 0\}$.

This curve has nodal singularities at the points $(1 : \epsilon_4^k : 0 : 0)$.

(iii) $C_3 = \{x^2 + y^2 = 0\}|_X = \{t^2 + z^3 + x^4z(2a - b) = 0\}$.

Unless $2a - b = 0$, C_3 is a smooth curve. When it is non-singular it is cuspidal.

(iv) $C_4 = \{x^2 - y^2 = 0\}|_X = \{t^2 + z^3 + x^4z(2a + b) = 0\}$.

As for C_3: When $2a + b = 0$ C_4 is cuspidal, otherwise it is smooth. □

6.1.3.16 Aut(X) = $\mathbb{D}_{12}$

Lemma 122.

$$\operatorname{lct}(X,G) = \operatorname{lct}_1(X,G) = 2.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x,y) + f_6(x,y) = 0,$$

with $f_4(x,y) = ax^2y^2$ and $f_6(x,y) = x^6 + y^6 + bx^3y^3$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [x, \epsilon_3 y, \epsilon_3 z, -t], g_2 = [y, x, z, t].$$

Generality conditions First observe that if $a = 0$, then the automorphism group of X jumps size to $\mathbb{Z}^2 \times \mathbb{Z}_{\bullet}^3\mathbb{D}_6$. Moreover, if $a = b = 0$ then Aut(X) = $\mathbb{Z}_{6\bullet}\mathbb{D}_{12}$.

Action of G **on** $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. First observe that if $a = 0$, then the automorphism group of X jumps size to $\mathbb{Z}^2 \times \mathbb{Z}_{\bullet}^3\mathbb{D}_6$.

G acts on $|-K_X| = \{\lambda x + \mu y = 0\}|_X$ with $(\lambda : \mu) \in \mathbb{P}^1$, yielding a 2-dim representation of G on $\mathbb{C}^2$. The generators g_1 and g_2 are given,respectively, by the matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon_3 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in this representation. We see that Γ_2 has Eigenvalues ± 1 and corresponding Eigenvectors $\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$. However

$$\Gamma_1 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \pm \epsilon_3 \end{bmatrix}.$$

Thus there is no common Eigenspace between the generators. Therefore the representation is irreducible, and the claim follows. □

Action of G **on** $|-2K_X|$

Claim. *The following four curves are the only* G*-invariant members of* $|-2K_X|$.

$$\begin{aligned}
C_1 &= \{xy=0\}|_X = \{t^2+z^3+x^6+y^6=0\},\\
C_2 &= \{z=0\}|_X = \{t^2+x^6+y^6+bx^3y^3=0\},\\
C_3 &= \{x^2+y^2=0\}|_X = \{t^2+z^3-ax^4z\pm bix^6=0\},\\
C_4 &= \{x^2-y^2=0\}|_X = \{t^2+z^3+ax^4z+(2\pm b)x^6=0\}.
\end{aligned}$$

Proof. Since for $\Delta \in |-2K_X|$, $\Delta = \{\lambda_0 x^2+\lambda_1 y^2+\lambda_2 xy+\lambda_3 z=0\}|_X$ with $(\lambda_0:\lambda_1:\lambda_2:\lambda_3)\in\mathbb{P}^3$, we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates x^2, y^2, xy and z). The generators g_1 and g_2 correspond to matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \epsilon_3^2 & 0 & 0\\ 0 & 0 & \epsilon_3 & 0\\ 0 & 0 & 0 & \epsilon_3\end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

respectively. We readily see that these split in to

$$\Gamma_1 = \begin{bmatrix}1\end{bmatrix}\oplus\begin{bmatrix}\epsilon_3^2\end{bmatrix}\oplus\begin{bmatrix}\epsilon_3\end{bmatrix}\oplus\begin{bmatrix}\epsilon_3\end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}.$$

Thus, the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^1$. This gives us four 1-dimensional irreducible subspaces corresponding to the G-invariant curves $C_1=\{xy=0\}|_X$, $C_2=\{z=0\}|_X$, $C_3=\{x^2+y^2=0\}|_X$ and $C_4=\{x^2-y^2=0\}|_X$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2\big(X,G\big)=2.$$

Proof. We examine each of the curves $C_1,\dots,C_4$ in turn and determine how bad their singularities can be.

(i) $C_1=\{xy=0\}|_X=\{t^2+z^3+x^6+y^6=0\}$.

Clearly this is a smooth curve on X.

(ii) $C_2 = \{z=0\}|_X = \{t^2 + x^6 + y^6 + bx^3y^3 = 0\}$.

This curve has nodal singularities at the points $(1:\pm1:0:0)$.

(iii) $C_3 = \{x^2 + y^2 = 0\}|_X = \{t^2 + z^3 - ax^4z \pm bix^6 = 0\}$.

If $a = b = 0$, C_3 is a cuspidal curve. If a and b are related by $4a\sqrt{\frac{a}{3}} \pm 6bi = 0$, then C_3 has nodal points at $(1:\pm i:\pm\sqrt{\frac{a}{3}}:0)$. For other values of a and b C_3 is non-singular.

(iv) $C_4 = \{x^2 - y^2 = 0\}|_X = \{t^2 + z^3 + ax^4z + (2\pm b)x^6 = 0\}$.

If $a = b = 0$, C_4 is a cuspidal curve. If a and b are related by $6(2\pm b) \pm 4ai\sqrt{\frac{a}{3}} = 0$, then C_4 has nodal points at $(1:\pm1:\pm i\sqrt{\frac{a}{3}}:0)$. For other values of a and b C_4 is non-singular. □

6.1.3.17 $\mathrm{Aut(X)} = \mathbb{D}_{16}$

Lemma 123.

$$\mathrm{lct}\big(X,G\big) = \mathrm{lct}_2\big(X,G\big) = \frac{5}{3}.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x,y) + f_6(x,y) = 0,$$

with $f_4(x,y) = ax^2y^2$ and $f_6(x,y) = xy(x^4 + y^4)$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [\epsilon_8 x, \epsilon_8^{-1} y, -z, it], g_2 = [y,x,z,t].$$

Generality conditions Observe that if $f_4(x,y) = 0$ then the automorphism group of X will jump to $\mathbb{Z}_3 \times \mathbb{Z}_2 . \mathbb{S}_4$, thus we require $a \neq 0$.

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. G acts on $|-K_X| = \{\lambda x + \mu y = 0\}|_X$ for some $(\lambda : \mu) \in \mathbb{P}^1$, yielding a 2-dim representation of G on $\mathbb{C}^2$. The generators g_1 and g_2 are given,respectively, by the matrices

$$\Gamma_1 = \begin{bmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in this representation. We see that Γ_2 has Eigenvalues ± 1 and corresponding Eigenvectors $\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$. However

$$\Gamma_1 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} \epsilon_8 \\ \pm\epsilon_8^{-1} \end{bmatrix}.$$

Thus there is no common Eigenspace between the generators. Therefore the representation is irreducible, and the claim follows. □

Action of G on $|-2K_X|$

Claim. *The following four curves are the only G-invariant members of $|-2K_X|$.*

$$\begin{aligned} C_1 &= \{xy = 0\}|_X = \{t^2 + z^3 = 0\}, \\ C_2 &= \{z = 0\}|_X = \{t^2 + xy(x^4 + y^4) = 0\}, \\ C_3 &= \{x^2 + y^2 = 0\}|_X = \{t^2 + z^3 - ax^4 z \pm 2ix^6 = 0\}, \\ C_4 &= \{x^2 - y^2 = 0\}|_X = \{t^2 + z^3 + ax^4 z \pm 2x^6 = 0\}. \end{aligned}$$

Proof. Since for $\Delta \in |-2K_X|$, $\Delta = \{\lambda_0 x^2 + \lambda_1 y^2 + \lambda_2 xy + \lambda_3 z = 0\}|_X$ with $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$, we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates x^2, y^2, xy and z). The generators g_1 and g_2 correspond to matrices

$$\Gamma_1 = \begin{bmatrix} \epsilon_8^2 & 0 & 0 & 0 \\ 0 & \epsilon_8^{-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively. We readily see that these split in to

$$\Gamma_1 = \begin{bmatrix}\epsilon_8^2\end{bmatrix} \oplus \begin{bmatrix}\epsilon_8^{-2}\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}-1\end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix}.$$

Thus, the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$. This gives us four 1-dimensional irreducible subspaces corresponding to the G-invariant curves $C_1 = \{xy = 0\}|_X$, $C_2 = \{z = 0\}|_X$, $C_3 = \{x^2 + y^2 = 0\}|_X$ and $C_4 = \{x^2 - y^2 = 0\}|_X$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\operatorname{lct}_2(X, G) = \frac{5}{3}.$$

Proof. We examine each of the curves $C_1, \dots, C_4$ in turn and determine how bad their singularities can be.

(i) $C_1 = \{xy = 0\}|_X = \{t^2 + z^3 = 0\}$.

This is a cuspidal curve on X.

(ii) $C_2 = \{z = 0\}|_X = \{t^2 + xy(x^4 + y^4) = 0\}$.

This curve has nodal singularities at the points $(1 : \epsilon_4^k : 0 : 0)$.

(iii) $C_3 = \{x^2 + y^2 = 0\}|_X = \{t^2 + z^3 - ax^4 z \pm 2ix^6 = 0\}$.

If $a = -3\epsilon_3^k$, then C_3 has nodal points at $(1 : \pm i : \pm\sqrt{\frac{a}{3}} : 0)$. For other values of a C_3 is non-singular.

(iv) $C_4 = \{x^2 - y^2 = 0\}|_X = \{t^2 + z^3 + ax^4 z + \pm 2x^6 = 0\}$.

If $a = 3i\epsilon_3^k$,then C_4 has nodal points at $(1 : \pm 1 : \pm i\sqrt{\frac{a}{3}} : 0)$. For other values of a C_4 is non-singular. □

6.1.3.18 $\mathrm{Aut(X)} = \mathbb{Z}_2 \bullet \mathbb{D}_{12}$

Lemma 124.

$$\operatorname{lct}(X, G) = \operatorname{lct}_2(X, G) = 2.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + z f_4(x, y) + f_6(x, y) = 0,$$

with $f_4(x, y) = ax^2y^2$ and $f_6(x, y) = x^6 + y^6$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [x, \epsilon_6 y, \epsilon_3^2 z, t], g_2 = [y, x, z, t], \beta.$$

Generality conditions Observe that if $a = 0$, then the automorphism group of X will be $\mathbb{Z}_6 \bullet \mathbb{D}_{12}$. Hence $a \neq 0$.

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. First observe that if $a = 0$, then the automorphism group of X will be $\mathbb{Z}_6 \bullet \mathbb{D}_{12}$. Hence $a \neq 0$.

G acts on $|-K_X| = \{\lambda x + \mu y = 0\}|_X$ with $(\lambda : \mu) \in \mathbb{P}^1$, yielding a 2-dim representation of G on $\mathbb{C}^2$. The generators g_1, g_2 and β are given,respectively, by the matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon_6 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \mathrm{Id}$$

in this representation. We see that Γ_2 has Eigenvalues ± 1 and corresponding Eigenvectors

$\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$. However

$$\Gamma_1 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \pm\epsilon_6 \end{bmatrix}.$$

Thus there is no common Eigenspace between the generators. Therefore the representation is irreducible, and the claim follows. □

Action of G on $|-2K_X|$

Claim. *The following four curves are the only G-invariant members of $|-2K_X|$.*

$$\begin{aligned}
C_1 &= \{xy = 0\}|_X = \{t^2 + z^3 + x^6 = 0\}, \\
C_2 &= \{z = 0\}|_X = \{t^2 + x^6 + y^6 = 0\}, \\
C_3 &= \{x^2 + y^2 = 0\}|_X = \{t^2 + z^3 - ax^4z - 2ix^6 = 0\}, \\
C_4 &= \{x^2 - y^2 = 0\}|_X = \{t^2 + z^3 + ax^4z + 2x^6 = 0\}.
\end{aligned}$$

Proof. Since for $\Delta \in |-2K_X|$, $\Delta = \{\lambda_0 x^2 + \lambda_1 y^2 + \lambda_2 xy + \lambda_3 z = 0\}|_X$ with $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$, we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates x^2, y^2, xy and z). The generators g_1, g_2 and β correspond to matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon_6^2 & 0 & 0 \\ 0 & 0 & \epsilon_6 & 0 \\ 0 & 0 & 0 & \epsilon_3^2 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \mathrm{Id},$$

respectively. We readily see that these split in to

$$\Gamma_1 = \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} \epsilon_3^2 \end{bmatrix} \oplus \begin{bmatrix} \epsilon_3 \end{bmatrix} \oplus \begin{bmatrix} \epsilon_3 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}.$$

Thus, the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$. This gives us four 1-dimensional irreducible subspaces corresponding to the G-invariant curves $C_1 = \{xy = 0\}|_X$, $C_2 = \{z = 0\}|_X$, $C_3 = \{x^2 + y^2 = 0\}|_X$ and $C_4 = \{x^2 - y^2 = 0\}|_X$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2\big(X,G\big)=2.$$

Proof. We examine each of the curves $C_1,\ldots,C_4$ in turn and determine how bad their singularities can be.

(i) $C_1=\{xy=0\}|_X=\{t^2+z^3+x^6=0\}$

Clearly this is a smooth curve on X.

(ii) $C_2=\{z=0\}|_X=\{t^2+x^6+y^6=0\}$.

This curve is non-singular on X.

(iii) $C_3=\{x^2+y^2=0\}|_X=\{t^2+z^3-ax^4z=0\}$.

If $a=0$, C_3 is a cuspidal curve — however if $a=0$, then we have a smooth curve in $|-K_X|^G$. For non-zero values of a, this is a smooth curve.

(iv) $C_4=\{x^2-y^2=0\}|_X=\{t^2+z^3+ax^4z+2x^6=0\}$.

If $a=3i\epsilon_3^k$, then C_4 has nodal singularities at the points $(1:\pm1:\pm\sqrt{\frac{a}{3}}:0)$. For other values of a and b C_4 is non-singular. $\square$

6.1.3.19 Aut(X) $=\mathbb{Z}_2\bullet\mathbb{A}_4$

Lemma 125.

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_2\big(X,G\big)=2.$$

Equation of surface and group action (binar tetrahedron group)

Equation of X:

$$t^2+z^3+zf_4(x,y)+f_6(x,y)=0,$$

with $f_4(x,y)=x^4+2i\sqrt{3}x^2y^2+y^4$ and $f_6(x,y)=xy(x^4-y^4)$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [ix, -iy, z, t], g_2 = [iy, ix, z, t], g_3 = [\frac{\epsilon_8^{-1}}{\sqrt{2}}(x+y), \frac{\epsilon_8}{\sqrt{2}}(-x+y), \epsilon_3 z, t].$$

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. G acts on $|-K_X| = \{\lambda x + \mu y = 0\}|_X$ for $(\lambda : \mu) \in \mathbb{P}^1$, yielding a 2-dim representation of G on $\mathbb{C}^2$. The generators g_1, g_2 are given,respectively, by the matrices

$$G_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \text{and} \quad G_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

in this representation. We see that G_2 has Eigenvalues $\pm i$ and corresponding Eigenvectors $\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$. However

$$G_1 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} i \\ \mp i \end{bmatrix}.$$

Thus there is no common Eigenspace between the generators. Therefore the representation is irreducible, and the claim follows. □

Action of G on $|-2K_X|$

Claim. *The only G-invariant member of $|-2K_X|$ is*

$$C = \{z = 0\}|_X = \{t^2 + xy(x^4 - y^4) = 0\}.$$

Proof. Since for $\Delta \in |-2K_X|$, $\Delta = \{\lambda_0 x^2 + \lambda_1 y^2 + \lambda_2 xy + \lambda_3 z = 0\}|_X$ with $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$, we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates

x^2, y^2, xy and z). The generators g_1, g_2 and g_3 correspond, respectively, to matrices

$$\Gamma_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma_3 = \begin{bmatrix} \frac{-i}{2} & \frac{-i}{2} & -i & 0 \\ \frac{i}{2} & \frac{i}{2} & -i & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}\epsilon_3 \end{bmatrix}.$$

We see that the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^3 \oplus \mathbb{C}^1$. This corresponds to a unique G-invariant curve in $|-2K_X|$, $C = \{z = 0\}|_X$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2(X, G) = 2.$$

Proof. By inspection, C is a smooth curve, hence result. □

6.1.3.20 Aut(X) $= \mathbb{Z}_3 \times \mathbb{D}_8$

Lemma 126.

$$\mathrm{lct}(X, G) = \mathrm{lct}_2(X, G) = \frac{5}{3}.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x, y) + f_6(x, y) = 0,$$

with $f_4(x, y) = 0$ and $f_6(x, y) = xy(x^4 + ax^2y^2 + y^4)$.

Here the group acts as it did for $\mathbb{D}_8$, the only difference being that we have taken specific coefficients to allow the automorphism group to enlarge (namely in Section 6.1.3.14; take $a = b = 0, c = 1$). Hence we have the following.

Lemma.

$$\mathrm{lct}_2(X, G) = \frac{5}{3}.$$

6.1.3.21 $\mathrm{Aut}(X) = \mathbb{Z}_2 \times \mathbb{Z}_3 {\scriptstyle\bullet} \mathbb{D}_6$

Lemma 127.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_2\big(X, G\big) = 2.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + z f_4(x, y) + f_6(x, y) = 0,$$

with $f_4(x, y) = 0$ and $f_6(x, y) = x^6 + a x^3 y^3 + y^6$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [x, y, \epsilon_3 z, t], g_2 = [x, \epsilon_3 y, z, t], g_3 = [y, x, z, t].$$

Generality conditions Observe that if $a = 0$, then the automorphism group of X will jump to $\mathbb{Z}_6 {\scriptstyle\bullet} \mathbb{D}_{12}$. Hence $a \neq 0$.

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. G acts on $|-K_X| = \{\lambda x + \mu y = 0\}|_X$ for $(\lambda : \mu) \in \mathbb{P}^1$, yielding a 2-dim representation of G on $\mathbb{C}^2$. The generators g_1, g_2 and g_3 are given,respectively, by the matrices

$$\Gamma_1 = \mathrm{Id}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon_3 \end{bmatrix} \quad \text{and} \quad \Gamma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in this representation. We see that Γ_3 has Eigenvalues ± 1 and corresponding Eigenvectors $\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$. However

$$\Gamma_2 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \pm \epsilon_3 \end{bmatrix}.$$

Thus there is no common Eigenspace between the generators. Therefore the representation is irreducible, and the claim follows. □

Action of G on $|-2K_X|$

Claim. *The following two curves are the only G-invariant members of $|-2K_X|$.*

$$\begin{aligned} C_1 &= \{xy = 0\}|_X, \\ C_2 &= \{z = 0\}|_X = \{t^2 + x^6 + y^6 = 0\}. \end{aligned}$$

Proof. Since for $\Delta \in |-2K_X|$, $\Delta = \{\lambda_0 x^2 + \lambda_1 y^2 + \lambda_2 xy + \lambda_3 z = 0\}|_X$ with $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$, we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates x^2, y^2, xy and z). The generators g_1, g_2 and g_3 correspond to matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon_3 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon_3^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively. We readily see that these split into

$$\Gamma_1 = \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} \epsilon_3 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon_3^2 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} = H_1 \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \quad \text{and}$$

$$\Gamma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} = H_2 \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}.$$

The matrix H_2 has Eigenvalues ± 1 and corresponding Eigenvectors $\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$. However,

$$H_1 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \pm \epsilon_3^2 \end{bmatrix}.$$

Therefore there is no common Eigenspace between H_1 and H_2.

Thus, the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^2 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$.

This gives us two 1-dimensional irreducible subspaces corresponding to the G-invariant curves $C_1 = \{xy = 0\}|_X$ and $C_2 = \{z = 0\}|_X$. $\square$

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2\big(X, G\big) = 2.$$

Proof. We examine each of the curves C_1, C_2 to find that C_1 is non-singular and C_2 has nodal singularities. $\square$

6.1.3.22 Aut(X) = $\mathbb{Z}_6 \bullet \mathbb{D}_{12}$

Lemma 128.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_2\big(X, G\big) = 2.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + zf_4(x, y) + f_6(x, y) = 0,$$

with $f_4(x, y) = 0$ and $f_6(x, y) = x^6 + y^6$.

Generators of Aut(X):

$$g_1 = [x, y, \epsilon_3 z, t], g_2 = [x, \epsilon_6 y, z, t], g_3 = [y, x, z, t].$$

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. G acts on $|-K_X| = \{\lambda x + \mu y = 0\}|_X$ with $(\lambda : \mu) \in \mathbb{P}^1$, yielding a 2-dim representation of

G on $\mathbb{C}^2$. The generators g_1, g_2 and g_3 are given,respectively, by the matrices

$$\Gamma_1 = \mathrm{Id}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon_6 \end{bmatrix} \quad \text{and} \quad \Gamma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in this representation. We see that Γ_3 has Eigenvalues ± 1 and corresponding Eigenvectors $\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$. However

$$\Gamma_2 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \pm\epsilon_6 \end{bmatrix}.$$

Thus there is no common Eigenspace between the generators. Therefore the representation is irreducible, and the claim follows. □

Action of G on $|-2K_X|$

Claim. *The following two curves are the only G-invariant members of $|-2K_X|$.*

$$\begin{aligned} C_1 &= \{xy = 0\}|_X, \\ C_2 &= \{z = 0\}|_X = \{t^2 + x^6 + y^6 = 0\}. \end{aligned}$$

Proof. Since for $\Delta \in |-2K_X|$, $\Delta = \{\lambda_0 x^2 + \lambda_1 y^2 + \lambda_2 xy + \lambda_3 z = 0\}|_X$ with $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$, we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates x^2, y^2, xy and z). The generators g_1, g_2 and g_3 correspond to matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon_3 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively. We readily see that these split into

$$\Gamma_1 = \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}\epsilon_3\end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix}1 & 0\\ 0 & \epsilon_3\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} = H_1 \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \quad \text{and}$$

$$\Gamma_3 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} = H_2 \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix}.$$

The matrix H_2 has Eigenvalues ± 1 and corresponding Eigenvectors $\begin{bmatrix}1 & \pm 1\end{bmatrix}^T$. However,

$$H_1 \begin{bmatrix}1\\ \pm 1\end{bmatrix} = \begin{bmatrix}1\\ \pm\epsilon_3\end{bmatrix}.$$

Therefore there is no common Eigenspace between H_1 and H_2.

Thus, the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^2 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$. This gives us two 1-dimensional irreducible subspaces corresponding to the G-invariant curves $C_1 = \{xy = 0\}|_X$ and $C_2 = \{z = 0\}|_X$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2(X, G) = 2.$$

Proof. On examination, we find that both curves C_1, C_2 are non-singular. □

6.1.3.23 Aut(X) $= \mathbb{Z}_3 \times \mathbb{Z}_2 . \mathbb{S}_4$

Lemma 129.

$$\mathrm{lct}(X, G) = \mathrm{lct}_2(X, G) = \frac{5}{3}.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^3 + z f_4(x, y) + f_6(x, y) = 0,$$

with $f_4(x,y)=0$ and $f_6(x,y)=xy(x^4-y^4)$.

Generators of $\mathrm{Aut}(X)$:

$$g_1=[\epsilon_8 x,\epsilon_8^{-1}y,-z,it], \qquad g_2=[y,x,-z,it],$$
$$g_3=\frac{1}{\sqrt{2}}[\epsilon_8^{-1}x+\epsilon_8^{-1}y,\epsilon_8^5 x+\epsilon_8 y,\sqrt{2}z,\sqrt{2}t], \quad g_4=[x,y,\epsilon_3 z,t].$$

Action of G on $|-K_X|$

Claim. $|-K_X|^G=\emptyset$.

Proof. G acts on $|-K_X|=\{\lambda x+\mu y=0\}|_X$ with $(\lambda:\mu)\in\mathbb{P}^1$, yielding a 2-dim representation of G on $\mathbb{C}^2$. The generators g_1 and g_2 are given,respectively, by the matrices

$$\Gamma_1=\begin{bmatrix}\epsilon_8 & 0\\ 0 & \epsilon_8^{-1}\end{bmatrix} \quad \text{and} \quad \Gamma_2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

in this representation. We see that Γ_2 has Eigenvalues ±1 and corresponding Eigenvectors $\begin{bmatrix}1 & \pm1\end{bmatrix}^T$. However

$$\Gamma_1\begin{bmatrix}1\\ \pm1\end{bmatrix}=\begin{bmatrix}\epsilon_8\\ \pm\epsilon_8^{-1}\end{bmatrix}.$$

Thus there is no common Eigenspace between even these two generators. Therefore the representation is irreducible, and the claim follows. □

Action of G on $|-2K_X|$

Claim. *The following two curves are the only G-invariant members of $|-2K_X|$.*

$$C_1 = \{xy=0\}|_X=\{t^2+z^3=0\},$$
$$C_2 = \{z=0\}|_X=\{t^2+xy(x^4-y^4)=0\}.$$

Proof. Since for $\Delta\in|-2K_X|$, $\Delta=\{\lambda_0x^2+\lambda_1y^2+\lambda_2xy+\lambda_3z=0\}|_X$ with $(\lambda_0:\lambda_1:\lambda_2:\lambda_3)\in\mathbb{P}^3$, we see that G acting on $|-2K_X|$ yields a 4-dim representation of G on $\mathbb{C}^4$ (with co-ordinates x^2,y^2,xy and z). The generators g_1, g_2, g_3 and g_4 correspond to matrices

$$\Gamma_1 = \begin{bmatrix} \epsilon_8^2 & 0 & 0 & 0 \\ 0 & \epsilon_8^{-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} \frac{1}{\sqrt{2}}\epsilon_8^{-1} & \frac{1}{\sqrt{2}}\epsilon_8^{-1} & 0 & 0 \\ \frac{1}{\sqrt{2}}\epsilon_8^{5} & \frac{1}{\sqrt{2}}\epsilon_8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and } \Gamma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon_3 \end{bmatrix},$$

respectively. We readily see that these split in to

$$\begin{aligned}
\Gamma_1 &= \begin{bmatrix}\epsilon_8^2\end{bmatrix} \oplus \begin{bmatrix}\epsilon_8^{-2}\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}-1\end{bmatrix}, \\
\Gamma_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}-1\end{bmatrix} = H_2 \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}-1\end{bmatrix}, \\
\Gamma_3 &= \begin{bmatrix} \frac{1}{\sqrt{2}}\epsilon_8^{-1} & \frac{1}{\sqrt{2}}\epsilon_8^{-1} \\ \frac{1}{\sqrt{2}}\epsilon_8^{5} & \frac{1}{\sqrt{2}}\epsilon_8 \end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} = H_3 \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix}, \\
\Gamma_4 &= \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}\epsilon_3\end{bmatrix}.
\end{aligned}$$

The matrix H_2 has Eigenvalues ± 1 and corresponding Eigenvectors $\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$. However,

$$H_3 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(\epsilon_8^{-1} \pm \epsilon_8^{-1}) \\ \frac{1}{\sqrt{2}}(\epsilon_8^{5} \pm \epsilon_8) \end{bmatrix}.$$

Therefore there is no common Eigenspace between H_2 and H_3.

Thus, the representation splits as a sum of irreducible sub-representations as $\mathbb{C}^2 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$. This gives us two 1-dimensional irreducible subspaces corresponding to the G-invariant curves $C_1 = \{xy = 0\}|_X$ and $C_2 = \{z = 0\}|_X$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\operatorname{lct}_2(X, G) = \frac{5}{3}.$$

Proof. On examination, we find that C_1 is cuspidal and C_2 is nodal. $\square$

6.2 Degree Two

6.2.1 Background

Let X be a del Pezzo surface of degree two. Then the anti-canonical linear system yields a degree two cover of the projective plane ramified in a smooth quartic plane curve $R \subset X$,

$$\psi : X \xrightarrow{2:1} \mathbb{P}^2,$$

as such, we may consider X to be a hyper-surface of degree four in $\mathbb{P}_{xyzt}(1,1,1,2)$. After a change of coordinates all such surfaces may be given by an equation of the form

$$t^2 + f_4(x,y,z) = 0,$$

where f_4 is a homogeneous degree four polynomial and $R = \{f_4 = 0\}|_X$ (see Proposition 77).

Let G be the full group of automorphisms Aut(X) of our surface X, then G is finite by Lemma 78 and always contains the subgroup $\mathbb{Z}_2$ generated by the Geiser involution that swaps the sheets of the double cover, ψ of $\mathbb{P}^2$ (see Section 5.1.4). Details of the possible automorphism groups realising minimal pairs (X,G) and their corresponding equations and generators can be found in [DI10].

6.2.2 General results

We answer here completely our Questions A and C by calculating the global log canonical thresholds of the G-surfaces (X,G) as G runs through all possible minimal automorphism groups. Let $G = \mathrm{Aut(X)}$ act minimally on our degree two del Pezzo G-surface X.

Lemma 130. $\mathrm{lct}(X,G) \leqslant 2$ *for all possible full automorphism groups* G.

Proof. It is enough to exhibit a G-invariant member of the bi-anti-canonical linear system $|-2K_X|$. Observe then that the ramification divisor of the double cover ψ given by $R = \{t = 0\}|_X = \{f_4(x,y,z) = 0\}$ is a G-invariant member of $|-2K_X|$. □

From Cheltsov (Theorem 33) we know that

$$\operatorname{lct}(X,I)=\begin{cases}\frac{3}{4} & \text{if } |-K_X| \text{ contains tacnodal curves}\\ \frac{5}{6} & \text{if } |-K_X| \text{ contains no tacnodal curves}\end{cases}$$

where I is the trivial group and X is any degree two smooth del Pezzo with prescribed anti-canonical linear system.

Thus from this and Lemma 130 above we see that

$$\frac{3}{4} \leqslant \operatorname{lct}(X,G) \leqslant 2.$$

Lemma 131. *Suppose that there exists a curve* $C \in |-K_X|^G$, *then* $\operatorname{lct}(X,G) = \operatorname{lct}_1(X,G)$.

Proof. Suppose that there exists $\lambda \in \mathbb{Q}$ such that $\operatorname{lct}(X,G) < \lambda < \operatorname{lct}_1(X,G) \leqslant 1$. Then there exists a G-invariant effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X,\lambda D)$ is not log canonical.

By Lemma 96, $\mathrm{LCS}(\mathrm{X},\lambda\mathrm{D})$ is zero-dimensional. Set $H = (\lambda-1)K_X$. Then $K_X + \lambda D + H \sim_{\mathbb{Q}} L = \mathcal{O}_X$ is Cartier and H is nef and big and we may apply Corollary 93. Whence $\mathrm{LCS}(\mathrm{X},\lambda\mathrm{D})$ consists of at most $h^0(X,\mathcal{O}_X) = 1$ point, P.

Since this point P is fixed under the action of G, it must belong to the ramification curve R of the double cover ψ of $\mathbb{P}^2$ by X (see Remark 80). Let $L \in |-K_X|$ be a curve such that $\psi(L)$ is a line in $\mathbb{P}^2$ tangent to $\psi(R)$ at the point $\psi(P)$. The curve L consists of at most two components and is G-invariant by construction. We may assume, by Convexity (Lemma 5), that L is not contained in the support of D. Since $\mathrm{mult}_P \lambda D > 1$, intersecting L and λD we obtain our contradiction:

$$2 > 2\lambda = L \cdot \lambda D \geqslant \mathrm{mult}_P L \cdot \mathrm{mult}_P \lambda D > \mathrm{mult}_P L > 2. \qquad \square$$

Theorem 132. *Suppose that* $|-K_X|^G = \emptyset$, *then* $\operatorname{lct}(X,G) = \operatorname{lct}_2(X,G) = 2$.

Proof. Firstly, by direct calculation (see proofs of Section 6.2.3) the only possibilities for minimal pairs (X,G) where $|-K_X|^G=\emptyset$ are

$$G=\mathbb{S}_4\times\mathbb{Z}_2,\ \big(\mathbb{Z}_2^4\rtimes\mathbb{S}_3\big)\times\mathbb{Z}_2\text{ or }\mathbb{PSL}_2(\mathbb{F}_7)\times\mathbb{Z}_2.$$

By Lemma 130, $|-2K_X|^G$ is non-empty. Suppose that there exists $\lambda\in\mathbb{Q}$ such that

$$\operatorname{lct}\big(X,G\big)<\lambda<\operatorname{lct}_2\big(X,G\big)\leqslant 2.$$

Then there exists a G-invariant effective $\mathbb{Q}$-divisor

$$D=\sum_{i=0}^{r}d_iD_i\equiv -K_X,$$

where $d_i\in\mathbb{Q}_+$ and D_i are prime divisors, such that the pair $\big(X,\lambda D\big)$ is not log canonical. There are two possibilities for the pair to fail to achieve log canonicity; either some component D_k of D has large coefficient d_k, or D has a point of high multiplicity.

By Lemma 96, $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ is zero-dimensional and by Corollary 93 consists of at most three points.

Suppose $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ consists of exactly one point. Then as this point must be fixed under the group action, it belongs to the ramification divisor R of the double cover ψ of $\mathbb{P}^2$ by X. Let $L\in|-K_X|$ be a curve such that $\psi(L)$ is a line in $\mathbb{P}^2$ tangent to $\psi(R)$ at the point $\psi(P)$. The curve L is then a G-invariant member of the anti-canonical linear system — a contradiction.

Suppose that $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ consists of two points, P_1 and P_2 say. These points must belong to the ramification curve of the double cover by Remark 80. By Lemma 94, P_1 and P_2 impose independent linear conditions on $H^0\Big(X,\mathcal{O}_X\big(-K_X\big)\Big)$ — hence $\psi(P_1)\neq\psi(P_2)$. Furthermore, by the same argument for the previous case where $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ consists of one point, the curve L_i whose image $\psi(L_i)\subset\mathbb{P}^2$ is a line tangent to $\psi(R)$ and passing through $\psi(P_i)$ is a G-invariant member of $|-K_X|$. The only escape is that the points are in an H-orbit, where $G=\mathbb{Z}^2\times H$. The action of $G=\mathbb{Z}_2\times H$ on X, where the copy of $\mathbb{Z}_2$ is generated by the Geiser involution associated with the swapping of the sheets of the double cover $\psi:X\longrightarrow\mathbb{P}^2$, induces an action of H on $\mathbb{P}^2$ and hence on $\psi(R)$. Since the action of the stabilisers $\mathrm{Stab}_H(P_i)\leqslant H$ is linear and faithful on the tangent spaces of R at P_i, T_{P_i} it follows that

$$\mathrm{Stab}_G(P_i) \hookrightarrow \mathbb{C}^*$$

and thus $\mathrm{Stab}_G(P_i)$ are cyclic groups for $i = 1,2,3$. However, by the Orbit-Stabiliser theorem, the orders for the stabilisers in the three cases for $H = \mathbb{S}_4, \mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$ or $\mathbb{PSL}_2(\mathbb{F}_7)$ are 2^3, 2^5 or $2^3 \cdot 7$, respectively and thus cannot be cyclic. Moreover, the groups $\mathbb{S}_4$ and $\mathbb{PSL}_2(\mathbb{F}_7)$ have no maximal subgroups of orders greater than 12, 24 respectively and hence are prevented from having stabilisers of orders 2.

Hence $\mathrm{LCS}\big(\mathrm{X}, \lambda \mathrm{D}\big)$ consists of three points, P_1, P_2 and P_3 that belong to the ramification curve of the double cover. By Lemma 94, these three points are distinct and by the same reasoning as above, no pair of points or single point may be fixed by the action of G. Thus, the group permutes the points and we have a G-orbit of length three. By the considering the action of H on R as for the two point orbit, we see that there cannont be orbits of length three. □

Theorem 133. *Let X be a general[3] smooth minimal del Pezzo G-surface of degree two with the prescribed automorphism group G, then*

$$\mathrm{lct}\big(X, \mathrm{Aut(X)}\big) = \begin{cases} 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_2, \\ \frac{3}{4} & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_{\mathrm{i}}, \textit{for } i = 6 \textit{ or } 18, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_2 \times \mathbb{Z}_2, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{S}_3 \times \mathbb{Z}_2, \\ \frac{3}{4} & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_2 \times \mathbb{Z}_6, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{D}_8 \times \mathbb{Z}_2, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \big(\mathbb{D}_8 \rtimes \mathbb{Z}_2\big) \times \mathbb{Z}_2, \\ 2 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{S}_4 \times \mathbb{Z}_2, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_4 \times \mathbb{A}_4 \times \mathbb{Z}_2, \\ 2 & \textit{if}\, \mathrm{Aut(X)} = \big(\mathbb{Z}_4^2 \rtimes \mathbb{S}_3\big) \times \mathbb{Z}_2, \\ 2 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}_2. \end{cases}$$

[3]The required generality is made explicit in restrictions on the parameters of the defining equations of X — that is 'general' means not on the list of Theorem 134.

The following theorem lists the group-invariant global log canonical thresholds for special cases of smooth minimal del Pezzo G-surfaces of degree two.

Theorem 134. *Let X be a smooth minimal del Pezzo G-surface of degree two, then*

$$\mathrm{lct}\big(X, \mathrm{Aut(X)}\big) = \begin{cases} \frac{3}{4} & \textit{if } \mathrm{Aut(X)} = \mathbb{Z}_2 \textit{ and } |-K_X| \textit{ contain cuspidal curves,} \\ \frac{5}{6} & \textit{if } \mathrm{Aut(X)} = \mathbb{Z}_2 \textit{ and } |-K_X| \textit{ contain tacnodal curves,} \\ \frac{3}{4} & \textit{if } \mathrm{Aut(X)} = \mathbb{Z}_2 \times \mathbb{Z}_2 \textit{ and } |-K_X|^G \textit{ contain cuspidal curves,} \\ \frac{5}{6} & \textit{if } \mathrm{Aut(X)} = \mathbb{Z}_2 \times \mathbb{Z}_2 \textit{ and } |-K_X|^G \textit{ contain tacnodal curves.} \end{cases}$$

6.2.3 Results for individual automorphism groups

Let X be a smooth minimal del Pezzo G-surface of degree two such that $G = \mathrm{Aut(X)}$ and x, y, z, t be homogeneous coordinates on $\mathbb{P}(1,1,1,2)$ with weights $1,1,1,2$, respectively. Denote the automorphism $\varphi : X \longrightarrow X$ mapping

$$(x : y : z : t) \mapsto \big(\varphi(x) : \varphi(y) : \varphi(z) : \varphi(t)\big) \quad \text{by} \quad \big[\varphi(x), \varphi(y), \varphi(z), \varphi(t)\big]$$

let $\epsilon_k = e^{\frac{2\pi i}{k}}$ be the k^{th} primitive root of unity and write γ for the Geiser involution $[x, y, z, -t]$. All notations are described in detail in Chapter 5.

6.2.3.1 Aut$(X) = \mathbb{Z}_2$

Lemma 135.

$$\mathrm{lct}\big(X, \mathbb{Z}_2\big) = \begin{cases} \frac{3}{4} & \textit{if } |-K_X|^G \textit{ contains tacnodal curves,} \\ \frac{5}{6} & \textit{if } |-K_X|^G \textit{ contains no tacnodal curves.} \end{cases}$$

Equation of surface and group action

Equation of X:

$$t^2 + f_4(x, y, z) = 0.$$

Generator of $\mathrm{Aut}(X)$ (Geiser involution):

$$\gamma = [x, y, z, -t].$$

Proof of Lemma 135. As the automorphism group of any smooth del Pezzo surface of degree two contains $\mathbb{Z}_2$ as a subgroup (Theorem 79), it follows that $\mathrm{lct}(X, \mathbb{Z}_2) = \mathrm{lct}(X, I)$, where I is the trivial group. □

6.2.3.2 $\mathbf{Aut}(X) = \mathbb{Z}_6$

Lemma 136.

$$\mathrm{lct}(X, G) = \mathrm{lct}_1(X, G) = \frac{3}{4}.$$

Equation of surface and group action

General Equation of X:

$$t^2 + xz^3 + f_4(x, y) = 0.$$

Here Aut(X) may belong to two different conjugacy classes:

(i) Equation of X:

$$t^2 + z^3 f_1(x, y) + f_4(x, y) = 0.$$

Generator of $\mathrm{Aut}(X)$:

$$g = [x, y, \epsilon_3 z, -t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains the G-invariant pencil

$$\mathcal{P} = \{\lambda x + \mu y = 0\}|_X,$$

for $(\lambda : \mu) \in \mathbb{P}^1$ and the curve

$$C_1 = \{z = 0\}|_X = \{t^2 + f_4(x, y) = 0\}.$$

Singularities of G-invariant curves in $|-K_X|$ To prove Lemma 136 for this conjugacy class it is enough to exhibit a G-invariant curve in $|-K_X|$ the has a tacnodal point. Suppose then that $\lambda_1 \neq 0$; $\lambda_1 = -1$ then $\mathcal{P} = \{t^2 + \alpha x z^3 + \beta x^4 = 0\}$ where α and β are functions of λ_0. Indeed, as $\lambda_0 \in \mathbb{C}$, we are free to choose λ_0 such that it is a root of α or β. On choosing a value of λ_0 such that $\alpha = 0$ and $\beta \neq 0$ we find that $\mathcal{P} = \{t^2 + \beta x^4 = 0\}$ — a curve with a tacnode.

(ii) Equation of X:

$$t^2 + x^4 + y^4 + z^3 x + \alpha x^2 y^2 = 0.$$

Generator of $\mathrm{Aut}(X)$:

$$g = [x, -y, \epsilon_3 z, -t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains the three G-invariant curves:

$$\begin{aligned}
C_1 &= \{x = 0\}|_X = \{t^2 + y^4 = 0\},\\
C_2 &= \{y = 0\}|_X = \{t^2 + x^4 + z^3 x = 0\},\\
C_3 &= \{z = 0\}|_X = \{t^2 + x^4 + y^4 + \alpha x^2 y^2 = 0\}.
\end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$ The curve C_1 is clearly tacnodal, hence $\mathrm{lct}(X, G) = \mathrm{lct}_1(X, G) = \frac{3}{4}$.

6.2.3.3 Aut$(X) = \mathbb{Z}_{18}$

Lemma 137.

$$\mathrm{lct}(X, G) = \mathrm{lct}_1(X, G) = \frac{3}{4}.$$

Equation of surface and group action

Equation of X:

$$t^2 + x^4 + x y^3 + y z^3 = 0.$$

Generator of Aut(X):

$$g = [x, \epsilon_3 y, \epsilon_9^2 z, -t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains the three G-invariant curves:

$$\begin{aligned} C_1 &= \{x = 0\}|_X = \{t^2 + yz^3 = 0\}, \\ C_2 &= \{y = 0\}|_X = \{t^2 + x^4 = 0\}, \\ C_3 &= \{z = 0\}|_X = \{t^2 + x^4 + xy^3 = 0\}. \end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$ The curve C_2 is clearly tacnodal, hence $\mathrm{lct}(X,G) = \mathrm{lct}_1(X,G) = \frac{3}{4}$.

6.2.3.4 Aut(X) = $\mathbb{Z}_2 \times \mathbb{Z}_2$

Lemma 138.

$$\mathrm{lct}(X,G) = \begin{cases} \frac{3}{4} & \textit{if} \quad f_4(x,y) \textit{ and } f_2(x,y) \textit{ have a shared root,} \\ \frac{3}{4} & \textit{if} \quad f_4(x,y) = 0, x^4 \textit{ or } y^4, \\ \frac{5}{6} & \textit{if } f_4(x,y) = x^3y \textit{ or } f_4(x,y) = xy^3, \\ 1 & \textit{otherwise.} \end{cases}$$

Equation of surface and group action

Equation of X:

$$t^2 + z^4 + z^2 f_2(x,y) + f_4(x,y) = 0.$$

Generators of Aut(X):

$$g_1 = [x, y, z, -t], g_2 = [x, y, -z, t].$$

Generality conditions The polynomials $f_2(x,y)$ and $f_4(x,y)$ are general such that the resulting surface X is smooth. Thus all the cases of Lemma 138 may occur.

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains the G-invariant pencil

$$\mathcal{P} = \{\lambda x + \mu y = 0\}|_X,$$

for $(\lambda : \mu) \in \mathbb{P}^1$ and the curve

$$C_1 = \{z = 0\}|_X = \{t^2 + f_4(x, y) = 0\}.$$

Singularities of G-invariant curves in $|-K_X|$ Let us first examine the pencil $\mathcal{P}$. Suppose that $\mu \neq 0; \mu = -1$, then $\mathcal{P} = \{t^2 + z^4 + \alpha x^2 z^2 + \beta x^4 = 0\}$ where α and β are functions of λ. We see that if $\alpha(\lambda) = \beta(\lambda)$ are identically zero, or if $f_2(x, y)$ and $f_4(x, y)$ share a common root then $\mathcal{P}$ contains a tacnodal curve. Otherwise, if one (or both) of α and β are non-zero then $\mathcal{P}$ contains a curves with no worse than nodal points. For the curve C_1, it is possible for it to be smooth, nodal, cuspidal or tacnodal dependant on $f_4(x, y)$; if $f_4(x, y) = x^4$ or $f_4(x, y) = y^4$, then C_1 has a tacnode; if $f_4(x, y) = x^3 y$ or $f_4(x, y) = x y^3$, then C_1 has a cusp; otherwise it has singularities no worse than nodes.

6.2.3.5 $\mathbf{Aut}(X) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lemma 139.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^4 + y^4 + x^4 + a x^2 y^2 + b x^2 z^2 + c y^2 z^2 = 0,$$

with $a \neq b \neq c$.

Generators of $\mathrm{Aut}(X)$:

$$\gamma = [x, y, z, -t], g_1 = [x, y, -z, t], g_3 = [x, -y, z, t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains the three G-invariant curves:

$$\begin{aligned}
C_1 &= \{x=0\}|_X = \{t^2+z^4+y^4+cy^2z^2=0\},\\
C_2 &= \{y=0\}|_X = \{t^2+z^4+x^4+bx^2z^2=0\},\\
C_3 &= \{z=0\}|_X = \{t^2+x^4+y^4+ax^2y^2=0\}.
\end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X,G\big)=1.$$

Proof. We see that for general values of a,b and c these curves are smooth. For C_1 (resp. C_2, C_3), if $c=\pm 2$ (resp. $b=\pm 2$, $a=\pm 2$) then it has a nodal point. □

6.2.3.6 Aut(X) = $\mathbb{S}_3 \times \mathbb{Z}_2$

Lemma 140.

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)=1.$$

Equation of surface and group action

Equation of X:

$$t^2+x^4+ax^2yz+x(y^3+z^3)+by^2z^2=0.$$

Generators of Aut(X):

$$g_1=[x,z,y,t], g_2=[x,\epsilon_3 y,\epsilon_3^{-1}z,t], \gamma=[x,y,z,-t].$$

Action of G on $|-K_X|$

Claim. *Let $C=\{x=0\}|_X=\{t^2+by^2z^2=0\}$. Then C is the only G-invariant member of the anti-canonical linear system.*

Proof. The action of G on $|-K_X| \cong \mathbb{C}^3$ splits into a sum of a one-dimensional and irreducible two-dimensional sub-representations. Indeed, the generators g_1, g_2 and γ correspond to matrices

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = I \oplus S_1, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon_3 & 0 \\ 0 & 0 & \epsilon_3^{-1} \end{bmatrix} = I \oplus S_2 \quad \text{and} \quad T = \mathrm{Id}.$$

However, S_1 has Eigenvalues ± 1 corresponding to Eigenvectors $\begin{bmatrix} 1 & \pm 1 \end{bmatrix}^T$ but

$$S_2 \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} \epsilon_3 \\ \pm\epsilon_3^{-1} \end{bmatrix}.$$

Hence result. □

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = 1.$$

Proof. If b is zero, then our surface X will be singular. Thus $b \neq 0$ and the curve C_1 has at worst nodal singularities. □

6.2.3.7 Aut$(X) = \mathbb{Z}_2 \times \mathbb{Z}_6$

Lemma 141.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = \frac{3}{4}.$$

Equation of surface and group action

Equation of X:

$$t^2 + x^4 + y^4 + xz^3 + ax^2y^2 = 0,$$

with $a \neq 0$.

Generators of $\mathrm{Aut}(X)$:

$$g = [x, \epsilon_6^3 y, \epsilon_6^2 z, t], \gamma = [x, y, z, -t].$$

Action of G on $|-K_X|$ By Lemma 30 and Remark 31 the anti-canonical linear system contains the three G-invariant curves:

$$\begin{aligned}
C_1 &= \{x = 0\}|_X = \{t^2 + y^4 = 0\},\\
C_2 &= \{y = 0\}|_X = \{t^2 + x^4 + xz^3 = 0\},\\
C_3 &= \{z = 0\}|_X = \{t^2 + x^4 + y^4 + ax^2y^2 = 0\}.
\end{aligned}$$

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = \frac{3}{4}.$$

Proof. The curve C_1 is tacnodal. □

6.2.3.8 Aut(X) = $\mathbb{D}_8 \times \mathbb{Z}_2$

Lemma 142.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^4 + x^4 + y^4 + ax^2y^2 + bz^2xy = 0,$$

with $a, b \neq 0$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [y, x, z, t], g_2 = [ix, -iy, z, t], \gamma = [x, y, z, -t].$$

Action of G on $|-K_X|$

Claim. *The only G-invariant element in $|-K_X|$ is the curve*

$$C = \{z=0\}|_X = \{t^2 + x^4 + y^4 + ax^2y^2 = 0\}.$$

Proof. By inspection of the generators of this action, we see that that the representation of G on $\mathbb{C}^3 \cong |-K_X|$ splits into a direct sum of a one-dimensional sub-representation and an irreducible two-dimensional sub-representation. Hence result. □

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = 1.$$

Proof. The curve C is smooth for general values of a. When $a=2$ we may write the equation for C as $t^2 + (x^2+y^2)^2$, which we can easily see has a pair of nodal points. □

6.2.3.9 $\mathrm{Aut(X)} = \big(\mathbb{D}_8 \rtimes \mathbb{Z}_2\big) \times \mathbb{Z}_2$

Lemma 143.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^4 + x^4 + ax^2y^2 + y^4 = 0,$$

with $a^2 \neq 0, -12, 4, 36$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [ix, iy, z, t], g_2 = [x, -y, z, t], g_3 = [y, x, z, t], \gamma = [x, y, z, -t].$$

Action of G on $|-K_X|$

Claim. *The only G-invariant element in $|-K_X|$ is the curve*

$$C = \{z = 0\}|_X = \{t^2 + x^4 + y^4 + ax^2y^2 = 0\}.$$

Proof. As for the previous group, by inspection of the generators of this action, we see that that the representation of G on $\mathbb{C}^3 \cong |-K_X|$ splits into a direct sum of a one-dimensional sub-representation and an irreducible two-dimensional sub-representation. Hence result. □

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = 1.$$

Proof. The curve C is smooth for general values of a. When $a = 2$ we may write the equation for C as $t^2 + (x^2 + y^2)^2$, which we can easily see has a pair of nodal points. □

6.2.3.10 $\mathrm{Aut(X)} = \mathbb{S}_4 \times \mathbb{Z}_2$

Lemma 144.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_2\big(X, G\big) = 2.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^4 + y^4 + x^4 + a(x^2y^2 + x^2z^2 + y^2z^2) = 0,$$

if $a = \frac{-1\pm\sqrt{7}}{2}$ or $a = 0$, then the automorphism group will be larger than $\mathbb{S}_4 \times \mathbb{Z}_2$. For $a = \frac{-1\pm\sqrt{7}}{2}$, $\mathrm{Aut(X)} = \mathbb{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}^2$ and for $a = 0$ $\mathrm{Aut(X)} = \mathbb{Z}^2 \times \mathrm{Z}_4^2 \rtimes \mathbb{S}_3$. Thus we require $a \neq 0$ or $\frac{-1\pm\sqrt{7}}{2}$.

Generators of $\mathrm{Aut}(X)$:

$$g_1 = [x, y, -z, t], \quad g_2 = [y, x, z, t], \quad g_3 = [x, z, y, t], \quad \gamma = [x, y, z, -t].$$

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. The claim follows from the fact that G acts on $\mathbb{C}^3 \cong |-K_X|$ by permutation and sign changes of the coordinates x, y and z. □

Action of G on $|-2K_X|$ On examining the action of G on $|-2K_X| = \langle t, x^2, y^2, z^2, xy, xz, yz\rangle$ we see that $|-2K_X|^G$ consists of two curves

$$C_1 = \{t=0\}|_X = \{z^4 + y^4 + x^4 + a(x^2y^2 + x^2z^2 + y^2z^2) = 0\},$$

(the ramification divisor of the double cover of $\mathbb{P}^2$) and

$$C_2 = \{x^2 + y^2 + z^2 = 0\}|_X = \{t^2 + a(z^4 - y^4) + (2+a)y^2z^2 = 0\}.$$

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2\big(X, G\big) = 2.$$

Proof. The curve C_1 is smooth, as it is the ramification divisor of the double cover of $\mathbb{P}^2$. For C_2, note that it has at most two singular points and so the pair $\big(X, C_2\big)$ is log canonical, by the proof of Theorem 132. □

6.2.3.11 $\mathrm{Aut(X)} = \mathbb{Z}_4 \bullet \mathbb{A}_4 \times \mathbb{Z}_2$

Lemma 145.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$t^2 + z^4 + x^4 + 2\sqrt{-3}x^2y^2 + y^4 = 0.$$

Generators of Aut(X):

$$g_1 = \left[\frac{x+iy}{1-i}, \frac{x-iy}{1-i}, \epsilon_6 z, t\right], \quad g_2 = [y, x, z, t],\ g_3 = [x, -y, z, t],$$
$$g_4 = [x, y, iz, t], \qquad \gamma = [x, y, z, -t].$$

Proof of Lemma 145. In this case we see that the automorphism group G contains the subgroup $H = \left(\mathbb{D}_8 \rtimes \mathbb{Z}_2\right) \times \mathbb{Z}_2$. From above, we know then that $\mathrm{lct}\left(X, H\right) = 1$. Thus to show the above Lemma it is enough to exhibit any G-invariant member of $|-K_X|$. Observe then that $C = \left\{z = 0\right\}|_X$ is a smooth G-invariant curve in $|-K_X|$. □

6.2.3.12 Aut(X) = $(\mathbb{Z}_4^2 \rtimes \mathbb{S}_3) \times \mathbb{Z}_2$

Lemma 146.

$$\mathrm{lct}\left(X, G\right) = \mathrm{lct}_2\left(X, G\right) = 2.$$

Equation of surface and group action

Equation of X:

$$t^2 + x^4 + y^4 + z^4 = 0.$$

Generators of Aut(X):

$$g_1 = [x, iy, z, t], g_2 = [y, x, z, t], g_3 = [x, z, y, t], \gamma = [x, y, z, -t].$$

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. Since $\mathbb{S}_3$ acts on the coordinates x, y and z a G-invariant divisor in $|-K_X|$ must be of the form $C = \{\lambda_0 x + \lambda_1 y + \lambda_2 z = 0\}$ with $(\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2$. However, $g_1(C) \neq C$. □

Action of G on $|-2K_X|$

Claim. *The only G-invariant curve in* $|-2K_X|$ *is* $T = \{t=0\}|_X = \{x^4+y^4+z^4=0\}$.

Proof. Since T is the ramification curve of the double cover of $\mathbb{P}^2$, it must be G-invariant. Let us try and construct another invariant curve in $|-2K_X|$. As before due to the subgroup $\mathbb{S}_3 = \langle g_2, g_3\rangle$ acting on $H^0\Big(X, \mathcal{O}_X\big(-2K_X\big)\Big) \cong \mathbb{C}^7$, any curve in $|-2K_X|$ must be of the form $C = \big\{\nu t + \lambda x^2 + \lambda y^2 + \lambda z^2 + \mu xy + \mu xz + \mu yz = 0\big\}$ with λ, μ not both zero (other wise we have the curve T). However, $g_1(C) \neq C$. □

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2\big(X, G\big) = 2.$$

Proof. The curve T is clearly smooth. □

6.2.3.13 Aut(X) = $\mathbb{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}_2$

Lemma 147.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_2\big(X, G\big) = 2.$$

Equation of surface and group action

Equation of X:

$$t^2 + x^3y + y^3z + z^3x = 0.$$

Generators of Aut(X):

$$\gamma = [x, y, z, -t], f = [\epsilon_7 x, \epsilon_7^2 y, \epsilon_7^4 z], g = [y, z, x], h,$$

where h is defined by the matrix

$$\frac{i}{7}\begin{bmatrix} \epsilon_7 - \epsilon_7^6 & \epsilon_7^2 - \epsilon_7^5 & \epsilon_7^4 - \epsilon_7^3 \\ \epsilon_7^2 - \epsilon_7^5 & \epsilon_7^4 - \epsilon_7^3 & \epsilon_7 - \epsilon_7^6 \\ \epsilon_7^4 - \epsilon_7^3 & \epsilon_7 - \epsilon_7^6 & \epsilon_7^2 - \epsilon_7^5 \end{bmatrix}.$$

Action of G on $|-K_X|$ It is easy to see that the subgroup generated by f and g leaves no divisor in $|-K_X|$ invariant, thus it follows that $|-K_X|^G = \emptyset$.

Action of G on $|-2K_X|$

Claim.

$$\mathrm{lct}_2\big(X, G\big) = 2.$$

Proof. On examining the representations of the generators of G on $\mathbb{C}^7 \cong |-2K_X|$ we find that: The representation of the subgroup generated by f splits as the direct sum $\mathbb{C} \oplus \mathbb{C} \cdots \oplus \mathbb{C}$ — acting identically on each copy of $\mathbb{C}$; the representation of the subgroup generated by g splits as $\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}$ — acting with the standard representation of cyclic group of order three on the copies of $\mathbb{C}^3$ and identically on the last copy of $\mathbb{C}$. This implies that the only curve in $|-2K_X|$ invariant under the subgroup of G generated by f and g is the (non-singular) ramification curve of the double cover of $\mathbb{P}^2$, $R = \{t = 0\}|_X$. Since this curve is also invariant under the actions of both γ and h we have that R is the only member of $|-2K_X|^G$ and $\mathrm{lct}_2\big(X, G\big) = 2$. □

Note: Alternatively, we could apply Theorem 97.

6.3 Degree Three — Cubic Surfaces

6.3.1 Background

Let X be a smooth del Pezzo surface of degree three, then X is isomorphic to a cubic surface (Proposition 77) and for generic X, Aut(X) = G is trivial (Proposition 79). For any X, G is finite by Lemma 78. Details of the possible automorphism groups realising minimal pairs (X,G) and their corresponding equations and generators can be found in [DI10] and for non-minimal pairs in [Hos97, Hos02].

6.3.2 General results

Let (X,G) be minimal with X and G as above. We answer here completely our Questions A and C of Section 3.2 by calculating the global log canonical thresholds of the G-surfaces (X,G) as (X,G) runs through all possible minimal pairs.

From Cheltsov (Theorem 33) we know that

$$\mathrm{lct}(X,I) = \begin{cases} \frac{2}{3} & \text{if } |-K_X| \text{ has curves with Eckardt points,} \\ \frac{3}{4} & \text{if } |-K_X| \text{ has curves with Eckardt points.} \end{cases}$$

where I is the trivial group and an Eckardt point is a point on X where three lines meet concurrently (see Definition 83).

Lemma 148. $\frac{2}{3} \leqslant \mathrm{lct}(X,G) \leqslant 4$ *for all possible automorphism groups* G.

Proof. From Theorem 33 we find the lower bound. The upper bound follows from Lemmata 166, 164 and the observation that for all $G \neq \mathbb{Z}_3(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4)$ or $\mathbb{S}_5$ the divisor $C = \{xyzt = 0\}|_X$ is a G-invariant member of $|-4K_X|^G$ and the log pair (X,C) is log canonical. □

Lemma 149. *Suppose that there exists a curve* $C \in |-K_X|^G$*, then* $\mathrm{lct}(X,G) = \mathrm{lct}_1(X,G)$.

Proof. Suppose that there exists $\lambda \in \mathbb{Q}$ such that $\mathrm{lct}(X,G) < \lambda < \mathrm{lct}_1(X,G) \leqslant 1$. Then there exists a G-invariant effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X,\lambda D)$ is not log canonical.

By Lemma 96, $\mathrm{LCS}(X, \lambda D)$ is zero-dimensional. Set $H = (\lambda - 1)K_X$. Then $K_X + \lambda D + H \sim_{\mathbb{Q}} L = \mathcal{O}_X$ is Cartier and H is nef and big and we may apply Corollary 93. Whence $\mathrm{LCS}(X, \lambda D)$ consists of at most $h^0(X, \mathcal{O}_X(L)) = 1$ point, P.

There is a birational morphism, $\pi : X \longrightarrow \mathbb{P}^2$ that is an isomorphism in a neighbourhood of P where $\pi(D) \equiv -K_{\mathbb{P}^2}$. Let L be a general line on $\mathbb{P}^2$, then

$$\mathrm{LCS}\big(\mathbb{P}^2, \pi(\lambda D) + L\big) = \big\{\pi(P) + L\big\}.$$

However, $-\big(K_{\mathbb{P}^2} + \pi(\lambda D) + L\big)$ is ample. Hence, by Shokurov Connectedness (Theorem 92), L and $\pi(P)$ are connected — contradiction. □

Suppose that (X, G) is minimal and that $|-K_X|^G = \emptyset$, then from Section 6.3.3 we see that the only possibilities for G are $\mathbb{S}_5$ or $\mathbb{Z}_3^2 \rtimes \mathbb{S}_4$.

Proposition 150 (Cheltsov [Che08])**.** *Let* $\mathrm{Aut}(X) = \mathbb{S}_5$ *or* $\mathbb{Z}_3^2 \rtimes \mathbb{S}_4$*, then*

$$\mathrm{lct}\big(X, \mathbb{S}_5\big) = \mathrm{lct}_2\big(X, \mathbb{S}_5\big) = 2,$$

or

$$\mathrm{lct}\big(X, \mathbb{Z}_3^2 \rtimes \mathbb{S}_4\big) = \mathrm{lct}_4\big(X, \mathbb{Z}_3^2 \rtimes \mathbb{S}_4\big) = 4,$$

respectively.

Proof. Let $G = \mathbb{S}_5$. We apply Theorem 97 with $k = 5$, $\xi = 2$, $r = 1$ and $H = -K_X$. Alternatively, see Lemmata 166 and 167.

Let $G = \mathbb{Z}_3^2 \rtimes \mathbb{S}_4$. In this case we cannot apply Theorem 97 as $h^0\Big(X, \mathcal{O}_X\big(-3K_X\big)\Big) = 19$ is too large. By inspection, we see that $|-K_X|^G = |-2K_X|^G = |-3K_X|^G = \emptyset$, and by Lemma 148 $\mathrm{lct}\big(X, G\big) \leqslant 4$.

Suppose, to seek a contradiction, that $\mathrm{lct}\big(X, G\big) < \lambda < 4$. Then there exists a G-invariant effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $\big(X, \lambda D\big)$ is not log canonical.

Lemma 96 and Corollary 93, imply that $\mathrm{LCS}(X, \lambda D)$ is zero-dimensional and consists of, at most, 19 points. Since the smallest G-orbit on X is the orbit of the 18 Eckardt points on X

and furthermore there are no orbits of 19 points since $19 \nmid 648 = |G|$, it follows that $\mathrm{LCS}\big(\mathrm{X}, \lambda \mathrm{D}\big)$ consists precisely of these 18 Eckardt points $P_1, \ldots, P_{18}$.

We may assume that $C \not\subseteq \mathrm{Supp}(D)$ by Lemma 5 and intersect C with D yielding

$$12 = C \cdot D \geqslant \sum_{i=1}^{18} \mathrm{mult}_{P_i}(C)\mathrm{mult}_{P_i}(D) \geqslant \sum_{i=1}^{18} 2\mathrm{mult}_{P_i}(D) \geqslant 36\mathrm{mult}_{P_i}(D),$$

that is,

$$\mathrm{mult}_{P_i}(D) \geqslant \frac{1}{3}. \tag{6.6}$$

Let $\pi : Y \to X$ be the blowup of X at the points $P_1, \ldots, P_{18}$ with exceptional divisors $E_i = \pi^{-1}(P_i)$ for $1 \leqslant i \leqslant 18$ and strict transforms denoted with bars. By taking the log pull-back of the pair $\big(X, \lambda D\big)$ under π we see that the pair

$$\Big(Y, \lambda \bar{D} + \sum_{i=1}^{18} \big(\lambda \mathrm{mult}_{P_i}(D) - 1\big) E_i\Big)$$

is not log canonical at some points $Q_1, \ldots Q_{18}$. From Remark 11 it follows that for $1 \leqslant i \leqslant 18$,

$$\mathrm{mult}_{Q_i}(\bar{D}) + \mathrm{mult}_{P_i}(D) > \frac{1}{2}. \tag{6.7}$$

Let Σ be a G-orbit of the point Q_k for some $1 \leqslant k \leqslant 18$, then $\Sigma \cap E_k \neq Q_k$ as the representation induced by the stabiliser of P_k on its tangent space is irreducible.

Intersecting E_k with $\bar{D}$ gives

$$\mathrm{mult}_{P_k}(D) = E_k \cdot \bar{D} \geqslant |\Sigma \cap E_k| \cdot \mathrm{mult}_{Q_k}(\bar{D}).$$

Together with Equations (6.6) and (6.7) this implies that $|\Sigma \cap E_k| = 1$ — a contradiction. □

We summarise these results in the following theorem.

Theorem 151. *Let X be a smooth minimal del Pezzo G-surface of degree three with the pre-*

scribed automorphism group G, then

$$\operatorname{lct}\big(X, Aut(X)\big) = \begin{cases} \frac{2}{3} & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_{\mathrm{i}}\ (i = 2,4,8), \\ \frac{2}{3} & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_2 \times \mathbb{Z}_2, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{S}_3\ \textit{or}\ \mathbb{S}_3 \times \mathbb{Z}_2, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{S}_4, \\ 1 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_3\big(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2\big)\ \textit{or}\ \mathbb{Z}_3\big(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4\big), \\ 2 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{S}_5, \\ 4 & \textit{if}\, \mathrm{Aut(X)} = \mathbb{Z}_3^3 \rtimes \mathbb{S}_4. \end{cases}$$

6.3.3 Results for individual automorphism groups

Let X be a smooth minimal cubic G-surface such that $G = \mathrm{Aut(X)}$ and x, y, z, t be homogeneous coordinates on $\mathbb{P}^3$. Denote the automorphism $\varphi : X \longrightarrow X$ mapping

$$(x : y : z : t) \mapsto \big(\varphi(x) : \varphi(y) : \varphi(z) : \varphi(t)\big) \quad \text{by} \quad \big[\varphi(x), \varphi(y), \varphi(z), \varphi(t)\big],$$

and let $\epsilon_k = e^{\frac{2\pi i}{k}}$ be the k^{th} primitive root of unity. All notations are described in detail in Chapter 5.

6.3.3.1 Aut$(X) = \mathbb{Z}_2$

Lemma 152.

$$\operatorname{lct}\big(X, G\big) = \operatorname{lct}_1\big(X, G\big) = \frac{2}{3}.$$

Here the generator g of our cyclic automorphism group may belong to two distinct conjugacy classes, $4A_1$ or $2A_1$ (in the notation of [DI10]).

(i) For $g \in 4A_1$:

Equation of surface and group action

Equation of X:

$$t^2 f_1(x, y, z) + x^3 + y^3 + z^3 + \alpha xyz = 0.$$

Generator of Aut(X):

$$g = [x, y, z, -t].$$

Generality conditions Observe that $f_1(x, y, z)$ cannot be identically zero as then G would act identically on X.

(ii) For $g \in 2A_1$:

Equation of surface and group action

Equation of X:

$$xz(z + \alpha t) + yt(z + \beta t) + x^3 + y^3 = 0.$$

Generator of Aut(X):

$$g = [x, y, -z, -t].$$

Proof of Lemma 152. It is enough in both cases to exhibit a G-invariant member of the anti-canonical linear system whose lct is $2/3$.

In the first case; we may write $f_1(x, y, z) = \lambda x + \mu y + \nu z$, with $(\lambda : \mu : \nu) \in \mathbb{P}^2$. Suppose that $\lambda \neq 0$, then we see that the curve $\{x = 0\}|_X$ splits into three distinct lines — that is X has an Eckardt point.

In the second case; consider the curve cut out of X by the hyperplane $H = \{z + t = 0\}$. This curve is G-invariant and splits into three lines making an Eckardt point. □

6.3.3.2 Aut(X) = $\mathbb{Z}_4$

Lemma 153.

$$\operatorname{lct}\big(X, G\big) = \operatorname{lct}_1\big(X, G\big) = \frac{2}{3}.$$

Again the generator g of our cyclic automorphism group may belong to two distinct conjugacy classes, $D_4(a_1)$ or $A_3 + A_1$ (in the notation of [DI10]).

(i) For $g \in D_4(a_1)$:

Equation of surface and group action

Equation of X:

$$t^2z + f_3(x,y) + z^2(x+\alpha y) = 0.$$

Generator of Aut(X):

$$g = [x, y, \epsilon_4^2 z, \epsilon_4 t].$$

(ii) For $g \in A_3 + A_1$:

Equation of surface and group action

Equation of X:

$$x^3 + xy^2 + yt^2 + yz^2 = 0.$$

Generator of Aut(X):

$$g = [x, \epsilon_4^2 y, \epsilon_4 z, \epsilon_4^3 t].$$

Proof of Lemma 153. As in the previous case where $G = \mathbb{Z}_2$; if we take hyperplane sections of X with $H = \{z = 0\}$ in the first case and $\{z + t = 0\}$ in the second we easily get that $\mathrm{lct}\big(X, G\big) \leqslant \frac{2}{3}$. Hence by Corollary 148 we are done. □

6.3.3.3 Aut$(X) = \mathbb{Z}_8$

Lemma 154.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = \frac{2}{3}.$$

Equation of surface and group action

Equation of X:

$$t^2y + z^2t + xy^2 + x^3 = 0.$$

Generator of Aut(X):

$$g = [x, \epsilon_8^4 y, \epsilon_8^3 z, \epsilon_8^2 t].$$

Proof of Lemma 154. Factoring the equation of X as $t(ty+z^2)+x(x-iy)(x+iy)=0$, we see that the hyperplane $H=\{t=0\}$ intersects X in an Eckardt point at $(0:0:1:0)$ and that $H|_X \in |-K_X|^G$. □

6.3.3.4 $\mathbf{Aut}(X)=\mathbb{Z}_2\times\mathbb{Z}_2$

Lemma 155.

$$\operatorname{lct}\big(X,G\big)=\operatorname{lct}_1\big(X,G\big)=\frac{2}{3}.$$

Equation of surface and group action

Equation of X:

$$t^2(x+y+az)+x^3+y^3+z^3+6bxyz=0.$$

Generators of Aut(X):

$$g_1=[x,y,z,-t],g_2=[y,x,z,t].$$

Generality conditions We require that $8b^3\neq -1$ to prevent the group from enlarging.

Proof of Lemma 155. By Proposition 85, we know that our cubic surface X has two Eckardt points — $(0:0:0:1)$ and $(1:-1:0:0)$ — corresponding to the isolated fixed points of the generators of G. Observe then that the curve cut out of X by the hyperplane $\{z=0\}$ is the union of three lines intersecting at the point $(0:0:0:1)$. □

6.3.3.5 $\mathbf{Aut}(X)=\mathbb{S}_3$

Lemma 156.

$$\operatorname{lct}\big(X,G\big)=\operatorname{lct}_1\big(X,G\big)=1.$$

Equation of surface and group action

Equation of X is:

$$x^3+y^3+azt(x+by)+z^3+t^3=0.$$

Generators of $\mathrm{Aut}(X)$:

$$\begin{tikzcd}
 & (x:y:\epsilon_3 z:\epsilon_3^2 t) & \\
(x:y:\epsilon_3^2 z:\epsilon_3 t) & (x:y:z:t) \arrow[u, mapsto, "g"] \arrow[l, mapsto, "g^2"'] \arrow[r, mapsto, "g_2"] \arrow[dl, mapsto, "g_0"'] \arrow[dr, mapsto, "g_1"] & (x:y:\epsilon_3^2 t:\epsilon_3 z) \\
(x:y:t:z) & & (x:y:\epsilon_3 t:\epsilon_3^2 z)
\end{tikzcd}$$

Generality conditions We require $a, b \in \mathbb{C}$ with $a \neq 0$ and $b^3 \neq -1$.

Action of G on $|-K_X|$

Claim 157. *For $0 \leqslant i \leqslant 2$, the points $P_i = (0:0:1:-\epsilon_3^i)$ are the only Eckardt points on X.*

Proof. We find the isolated fixed points of the automorphisms g, g^2, g_0, g_1, g_2 then apply Proposition 85. From the definitions of the maps above we see that g and g^2 fix the line $\mathcal{M} = \{z = t = 0\}$ and the points $R_1 = (0:0:0:1)$, $R_2 = (0:0:1:0)$ and for $i = 0, 1, 2$ the g_i fix planes $\Pi_i = \{t - \epsilon_3^i z = 0\}$ and points $P_i = (0:0:1:-\epsilon_3^i)$. Thus the only isolated fixed points we find are a P_i for each g_i. The proposition tells us that each of the Eckardt points on X corresponds to an isolated fixed point of an involution and we are done. □

It is clear from the maps g, g^2, g_0, g_1, g_2 that our $\mathbb{S}_3$ action here has a representation on $\mathbb{C}^4$ that decomposes as the direct sum $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^2$. The representations on first two $\mathbb{C}^1$ are the identity map and on the third factor, the standard two dimensional representation of $\mathbb{S}_3$.

Observe then that $\mathcal{L} = \{x = y = 0\}$ is an $\mathbb{S}_3$-invariant line not contained in the surface X. Indeed, from the equation of X or from the fact that P_1, P_2, P_3 lie on X and Proposition 86 (a line passing through three Eckardt points may not be contained in the surface) we see that $\mathcal{L}$ is not contained in the surface. That $\mathcal{L}$ is $\mathbb{S}_3$-invariant follows from the previous paragraph.This proves the following Claim.

Claim. *Let D be an effective $\mathbb{S}_3$-invariant $\mathbb{Q}$-divisor in the anti-canonical linear system $|-K_X|$. Then $D = H \cap X$ where $H \in \mathcal{H} = \{\lambda x + \mu y = 0\}$ is a member of the pencil of planes through the line $\mathcal{L}$ where $(\lambda : \mu) \in \mathbb{P}^2$.*

Singularities of G-invariant curves in $|-K_X|$

Claim 158. *The curves $\mathcal{C} = \mathcal{H} \cap X$ have, at worst, nodal singularities.*

Proof. Let H be a member of $\mathcal{H} = \{\lambda x + \mu y = 0\}$ and assume $\lambda \neq 0$. Then we may write

$$\mathcal{C} = H \cap X = \{y^3(1-\mu^3) + ayzt(b-\mu) + z^3 + t^3 = 0 : a \neq 0, b^3 \neq -1\}.$$

Looking at the partial derivatives

$$\begin{aligned} \frac{\partial \mathcal{C}}{\partial y} &= 3y^2(1-\mu^3) + azt(b-\mu), \\ \frac{\partial \mathcal{C}}{\partial z} &= ayt(b-\mu) + 3z^2, \\ \frac{\partial \mathcal{C}}{\partial t} &= ayz(b-\mu) + 3t^2, \end{aligned}$$

we see that the only possible singular points are

$$P_k = \left(\frac{3\mu\epsilon_3^{2k}}{a(b-\mu)} : -\frac{3\epsilon_3^{2k}}{a(b-\mu)} : 1 : \epsilon_3^k\right),$$

for $k = 0, 1, 2$ where μ satisfies

$$F(\mu) = 27(1-\mu^3) + a^3(b-\mu)^3 = 0$$

with $b \neq \mu$ and the points

$$Q_m = (-\epsilon_3^m : 1 : 0 : 0),$$

where ϵ_3 is the primitive cube root of unity.

Observe that, on the chart where $z \neq 0$, if $b = \mu$ then the curve $\mathcal{C}$ is non-singular here. Thus we may assume that $b \neq \mu$.

The Hessian matrix for $\mathcal{C}$ on this chart is:

$$\mathcal{A} = \begin{pmatrix} 6y(1-\mu^3) & a(b-\mu) \\ a(b-\mu) & 6t \end{pmatrix}.$$

The determinant of $\mathcal{A}$ evaluated at the singular points P_1, P_2, P_3 we found above yields the equation;

$$-108(1-\mu^3) = a^3(b-\mu),$$

where μ must satisfy $F(\mu)$. Putting this together we see that $\det\mathcal{A} = 0$ if, and only if, $\mu^3 = 1$. However this holds only when $b = \mu$ (since $a \neq 0$). Thus, the determinant of $\mathcal{A}$ is always non-zero on this chart. It follows that the points P_1, P_2, P_3 are nodal points.

Next, for $m = 0, 1, 2$, the three points $Q_m = (-\epsilon_3^m : 1 : 0 : 0)$ lie on the chart where $y \neq 0$. Here $z = 0$ implies that $\mu^3 = 1$. The equation of $\mathcal{C}$ is then;

$$azt(b-\mu) + z^3 + t^3 = 0.$$

Observe that if $b = \mu$ then the points Q_0, Q_1, Q_2 are three more Eckardt points on our surface. This would contradict Claim 157, so that $b \neq \mu$ on this chart also. As before, we examine the Hessian:

$$\mathcal{A} = \begin{pmatrix} 6z & a(b-\mu) \\ a(b-\mu) & 6t \end{pmatrix}.$$

The determinant of $\mathcal{A}$ restricted to the points Q_0, Q_1, Q_2 may only be zero if $a = 0$ or $b = \mu$. Hence, all the singular points Q_0, Q_1, Q_2 are nodes.

Alternatively, we may observe that for a fixed μ; $\mathcal{C}$ is a plane cubic curve with three singular points on it. Thus the only possibility is that $\mathcal{C}$ is the sum of three lines intersecting in three distinct points and forming a triangle. $\square$

Corollary 159.

$$\operatorname{lct}_1\big(X, \mathbb{S}_3\big) = 1.$$

Proof. Suppose that $\operatorname{lct}_1\big(X, \mathbb{S}_3\big) < 1$, then there exists an effective $\mathbb{S}_3$-invariant $\mathbb{Q}$-divisor, $B \in |-K_X|$ such that the log pair (X, B) is not lc. This, however, contradicts Claim 158. □

6.3.3.6 $\mathbf{Aut}(X) = \mathbb{S}_3 \times \mathbb{Z}_2$

Lemma 160.

$$\operatorname{lct}\big(X, G\big) = \operatorname{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$x^3 + y^3 + azt(x + y) + z^3 + t^3 = 0.$$

The difference between the surfaces we saw in Section 6.3.3.5 and the ones we'll look at here is an additional $\mathbb{Z}_2$-action. This involution swaps the coordinates x and y.

Generality conditions We relax our conditions from the previous section to allow the extra $\mathbb{Z}_2$-action. More explicitly, we require $a, b \in \mathbb{C}$ with $a \neq 0$.

Proof of Lemma 160. Observe that the introduction of this extra involution does not alter the log canonical threshold. Indeed it is easy to see, as we observed above, that the line $\mathcal{L} = \{x = y = 0\}$ is invariant under this larger automorphism action. As in the previous case we see that the analogues of Claim 158 and Corollary 159 are true in this case too. □

6.3.3.7 $\mathbf{Aut}(X) = \mathbb{S}_4$

Lemma 161.

$$\operatorname{lct}\big(X, G\big) = \operatorname{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action The equation of a cubic surface with automorphism group $\mathbb{S}_4$ and the action of $\mathbb{S}_4$ is described neatly in [DI10] (Section 6.5), which we paraphrase here.

$\mathbb{S}_4$ has four subgroups isomorphic to $\mathbb{S}_3$, each pair of which share a common element of order two. From Propositions 85 and 86, we see that this corresponds to four lines, $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$, in $\mathbb{P}^3$ each with three Eckardt points (one on each line is at infinity) and each pair meeting once in an Eckardt point, $P_1, \ldots, P_6$. Thus the four lines must form a quadrangle in the plane. Also observe that since each of the three diagonals, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, contains only two Eckardt points, these three lines must be contained in the surface. This is depicted in Figure 6.3.3.7.

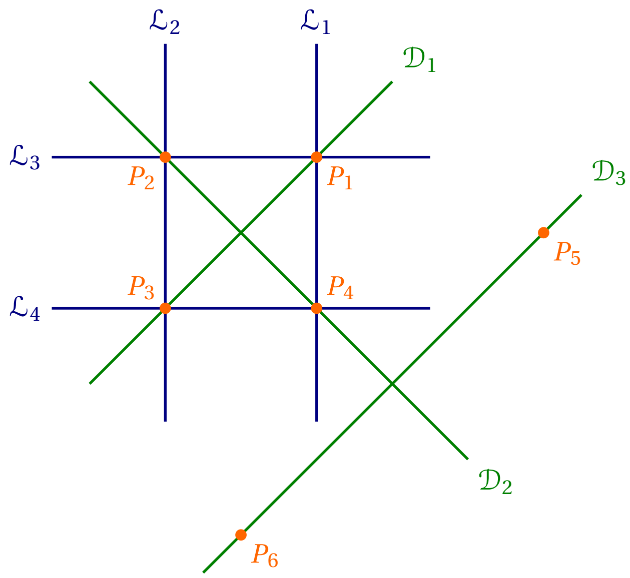


Figure 6.5: Lines in $\mathbb{P}^3$ and the $\mathbb{S}_4$-cubic surface.

Taking the equations of this plane to be $x = 0$ and the equations of the diagonals to be $x = y = 0$, $x = z = 0$ and $x = t = 0$. We see that the group acts by permutation of the coordinates y, z, t and multiplication by ± 1. The equation of our surface X is then

$$x^3 + x(y^2 + z^2 + t^2) + ayzt = 0.$$

Generality conditions We require for the variable a that $9a^3 \neq 8a$.

Action of G on $|-K_X|$

Claim. *$C = \{x = 0\}$ is the only $\mathbb{S}_4$-invariant hyperplane in $\mathbb{P}^3$.*

Proof. Let $C = \left\{\alpha x + \beta y + \gamma z + \delta t = 0 \mid (\alpha : \beta : \gamma : \delta) \in \mathbb{P}^3\right\}$ be a $\mathbb{S}_4$-invariant hyperplane and let $\sigma \in \mathbb{S}_4$. Then $\sigma C = \{\alpha x \pm \beta\sigma(y) \pm \gamma\sigma(z) \pm \delta\sigma(t) = 0\} = C$. Hence $C = \{x = 0\}$. □

Corollary 162. *The only $\mathbb{S}_4$-invariant effective $\mathbb{Q}$-divisor in the anti-canonical linear system $|K_X|$ is $C|_X$.*

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = 1.$$

Proof. By inspection, the curve $C|_X = \{x = 0\}|_X$ has no worse than nodal singularities. □

6.3.3.8 $\mathbf{Aut}(X) = \mathbb{Z}_3\big(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2\big)$

Lemma 163.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$x^3 + y^3 + z^3 + t^3 + 6ayzt = 0.$$

Generators of $\mathrm{Aut}(X)$:

$$g = [\epsilon_3 x, y, z, t], \tau = [x, y, \epsilon_3 z, \epsilon_3^2 t], h_1 = [x, z, t, y], h_2 = [x, z, y, t].$$

Generality conditions For special values of a, there may be further automorphisms of orders 4 or 6. When $1 - 20a^3 - 8a^6 = 0$ the extra automorphism of order four is given by

$$\xi = [x, y + z + t, y + \epsilon z + \epsilon^2 t, y + \epsilon^2 z + \epsilon t],$$

and the automorphism group grows to $\mathbb{Z}_3\big(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4\big)$ — see Section 6.3.3.8. When $a(a^3-1)=0$ the surface is projectively equivalent to surfaces with automorphism groups $\mathbb{Z}_3^3 \rtimes \mathbb{S}_4$ — see Section 6.3.3.11.

Thus we require that $a(a^3-1) \neq 0$ and $20a^3 + 8a^6 \neq 1$ here.

Action of G on $|-K_X|$ Observe that X has no sub-varieties fixed by the action of G. Indeed, looking at each individual automorphism; g fixes only the plane $H = \{x=0\}$ and the point $(1:0:0:0)$, τ fixes only the line $\mathcal{M} = \{z = t = 0\}$ and the points $(0:0:0:1)$ and $(0:0:1:0)$. The action of $\mathbb{S}_3$ on the coordinates y, z and t fixes only the line $\mathcal{N} = \{y = z = t = 1\}$ and the point $(1:0:0:0)$. Thus, we see that the only fixed sub-variety of the action G is the point $(1:0:0:0)$, which does not belong to X.

Claim. *Let C be an effective G-invariant $\mathbb{Q}$-divisor in the anti-canonical linear system $|-K_X|$. Then $C = \{x=0\}|_X$.*

Proof. The representation of G splits into two irreducible factors, $\mathbb{C}^1 \oplus \mathbb{C}^3$, with the first factor corresponding to the identity action on the coordinate x. Whence, the only G-invariant subspace of $\mathbb{P}^3$ is $\{x=0\}$ and thus, the only G-invariant divisor in the anti-canonical linear system is $D = \{x=0\}|_X$. □

Singularities of G-invariant curves in $|-K_X|$

Claim.

$$\mathrm{lct}_1\big(X, G\big) = 1.$$

Proof. By inspection, the curve $C = \{x=0\}|_X$ is smooth. □

6.3.3.9 $\mathbf{Aut}(X) = \mathbb{Z}_3\big(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4\big)$

Lemma 164.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_1\big(X, G\big) = 1.$$

Equation of surface and group action

Equation of X:

$$x^3 + y^3 + z^3 + t^3 + 6ayzt = 0.$$

Generators of $\mathrm{Aut}(X)$:

$$g = [\epsilon_3 x, y, z, t],\ \tau = [x, y, \epsilon_3 z, \epsilon_3^2 t],\ h_1 = [x, z, t, y],$$
$$h_2 = [x, z, y, t],\ \xi = [x, y + z + t, y + \epsilon_3 z + \epsilon_3^2 t, y + \epsilon_3^2 z + \epsilon_3 t].$$

Generality conditions We wish for the automorphism group to jump in size, namely that $1 - 20a^3 - 8a^6 = 0$ to allow the extra symmetry embodied in ξ. However we must restrict the automorphism group from jumping in size further to $\mathbb{Z}_3^3 \rtimes \mathbb{S}_4$, thus we also require that $a(a^3 - 1) \neq 0$.

Proof of Lemma 164. Word-for-word the two claims of the previous section (Section 6.3.3.8) are true in this case too. □

6.3.3.10 Aut$(X) = \mathbb{S}_5$

Lemma 165.

$$\mathrm{lct}\big(X, G\big) = \mathrm{lct}_2\big(X, G\big) = 2.$$

Equation of surface and group action Equation of X (Clebsch cubic):

$$x^2 y + xz^2 + zt^2 + tx^2 = 0.$$

X is isomorphic to a complete intersection in $\mathbb{P}^4$ with homogeneous coordinates $x_0, \ldots, x_4$ given by equations

$$\sum_{i=0}^{4} x_i^3 = \sum_{i=0}^{4} x_i = 0,$$

and here the action of $\mathbb{S}_5$ is realised as the standard representation on $x_0, \ldots, x_4$.

Lemma 166.

$$\mathrm{lct}\big(X,G\big) \leqslant \mathrm{lct}_2\big(X,G\big) \leqslant 2.$$

Proof. Observe that the divisor cut out by the equations $\sum_{i=0}^{4} x_i^2 = 0 \subset \mathbb{P}^4$ is a G-invariant member of $|-2K_X|$. □

See Proposition 150 or the following Lemma.

Lemma 167.

$$\mathrm{lct}\big(X,G\big) = 2.$$

Proof. Suppose that there exists $\lambda \in \mathbb{Q}$ such that $\mathrm{lct}\big(X,G\big) < \lambda < 2$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $\big(X,\lambda D\big)$ is not log canonical.

If $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ is zero-dimensional, then by Lemma 93 (with $H = (\lambda-2)K_X$) it consists of at most four points. However, the representation of $\mathbb{S}_5$ on $|-K_X|$ is irreducible and $\mathbb{S}_5$ has no orbits of length less than 5.

Thus $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ is not zero-dimensional and we may write $D = mC + \Omega$ where $\lambda m > 1$ and $C \in |-\xi K_X|^G$ for some $\xi \in \mathbb{Z}_{>0}$. Following the proof of Lemma 96, we conclude that $\xi = 1$; that is, C is an irreducible G-invariant curve in the anti-canonical linear system — impossible. □

6.3.3.11 Aut$(X) = \mathbb{Z}_3^3 \rtimes \mathbb{S}_4$

Lemma 168.

$$\mathrm{lct}\big(X,G\big) = \mathrm{lct}_4\big(X,G\big) = 4.$$

Equation of surface and group action Equation of X (Fermat Cubic):

$$x^3 + y^3 + z^3 + t^3 = 0.$$

The group $\mathbb{Z}_3^3 \rtimes \mathbb{S}_4$ acts in the obvious way — the standard representation of $\mathbb{S}_4$ on x, y, z and t and acting by multiplication by cube roots of unity.

For proof of the above Lemma, see Proposition 150.

6.4 Degree Four

6.4.1 Background

Let X be a smooth del Pezzo surface of degree four. Then by Proposition 77, X is a complete intersection of two quadrics in $\mathbb{P}^4$ with homogeneous coordinates $x_0, \ldots, x_4$ and can be given by the equations

$$\sum_{i=0}^{4} x_i^2 = \sum_{i=0}^{4} \alpha_i x_i^2 = 0$$

with $\alpha_i \neq \alpha_j$ for $i \neq j$ (see, for example, [Rei72, Proposition 2.1]).

Let G be the group of automorphisms Aut(X) of our surface X, then G is finite by Lemma 78. We answer here completely Questions A and C of Section 3.2. Details of the possible groups acting regularly on X such that the pairs (X,G) are minimal can be found in [DI10] (cf. [Hos96]).

Remark 169. By Proposition 79, the full automorphism group of a degree four del Pezzo surface always contains the subgroup $\mathbb{Z}_2^4$ generated by $s_i : x_i \mapsto -x_i$, for $i \in \{1, \ldots, 4\}$ (we get s_0 for free via the $\mathbb{C}^*$ action on $\mathbb{P}^4$).

6.4.2 General results

Let (X,G) be minimal with X and G as above. From Cheltsov (Theorem 33) we know that

$$\mathrm{lct}(X,I) = \frac{2}{3},$$

where I is the trivial group.

Lemma 170. *For any del Pezzo surface of degree four,*

$$\mathrm{lct}(X, \mathrm{Aut(X)}) \geqslant 1$$

Proof. This follows immediately from Claim 175 and Remark 169. □

Lemma 171. *Suppose that there exists $C \in |-K_X|^G$. Then* $\mathrm{lct}(X,G) = \mathrm{lct}_1(X,G) = 1$.

Proof. It follows from the assumption that there exists $C \in |-K_X|^G$ that $\mathrm{lct}\big(X,G\big) \leqslant 1$, together with Lemma 170 we are done. For a direct proof see below.

Suppose that there exists $\lambda \in \mathbb{Q}$ such that $\mathrm{lct}\big(X,G\big) < \lambda < \mathrm{lct}_1\big(X,G\big) = 1$. Then there is an effective G-invariant $\mathbb{Q}$-divisor $D = \sum_{i=1}^r d_i D_i \equiv -K_X$ on X such that the log pair $\big(X, \lambda D\big)$ is not log canonical.

Suppose that $\mathrm{LCS}\big(\mathrm{X}, \lambda \mathrm{D}\big)$ is not zero dimensional. Then we may assume that $\lambda d_1 > 1$. The intersection

$$4 = D \cdot (-K_X) \geqslant d_1 D_1 \cdot (-K_X) > \deg(D_1) \geqslant 1$$

shows that the curve D_1 has degree one, two or three. Let $H|_X = R + D_1$ be a hyperplane section of X passing through the curve D_1, then

$$3 > 3\lambda \geqslant \lambda(4 - \deg(D_1)) = R \cdot \lambda D \geqslant \lambda d_1 (R \cdot D_1) > R \cdot D_1 \geqslant 3.$$

Hence $\mathrm{LCS}\big(\mathrm{X}, \lambda \mathrm{D}\big)$ is zero dimensional and by Corollary 93 consists of at most one point, P.

There is a birational morphism $\pi : X \longrightarrow \mathbb{P}^2$ that is an isomorphism in a neighbourhood of the point P and $\pi(D) \equiv -K_{\mathbb{P}^2}$. Take a general line L on $\mathbb{P}^2$, then

$$\mathrm{LCS}\Big(\mathbb{P}^2, \pi\big(\lambda \mathrm{D}\big) + \mathrm{L}\Big) = \big\{\pi(\mathrm{P}) \cup \mathrm{L}\big\}$$

and $-\Big(K_{\mathbb{P}^2} + \pi\big(\lambda D\big) + L\Big)$ is ample. However, $\mathrm{LCS}\Big(\mathbb{P}^2, \pi\big(\lambda \mathrm{D}\big) + \mathrm{L}\Big)$ is not connected and this contradicts the Shokurov Connectedness Theorem (Theorem 92). □

Lemma 172. *Suppose that* $|-K_X|^G = \emptyset$. *Then* $\mathrm{lct}\big(X,G\big) = \mathrm{lct}_2\big(X,G\big) = 2$.

Proof. Firstly, observe that from Section 6.1.3 it follows that the only possibilities for G such that $\big(X,G\big)$ is minimal and $|-K_X|^G = \emptyset$ are

$$G = \mathbb{Z}_2^4 \rtimes \mathbb{S}_3 \text{ or } \mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}.$$

Suppose that there exists $\lambda \in \mathbb{Q}$ such that $\mathrm{lct}\big(X,G\big) < \lambda < \mathrm{lct}_2\big(X,G\big) = 2$. Then there is an effective $\mathbb{Q}$-divisor $D = \sum_{i=0}^r d_i D_i \equiv -K_X$, where $d_i \in \mathbb{Q}_{>0}$ and the D_i are prime Weil divisors such that the log pair $\left(X, \lambda D\right)$ is not log canonical.

Suppose that $\mathrm{LCS}\left(X, \lambda D\right)$ is not zero-dimensional. Then, without loss of generality, $\lambda d_1 > 1$. Write $D = d_1(\Delta_1 + \ldots + \Delta_k) + \Omega$, where $\Delta_1 + \ldots + \Delta_k$ is a G-orbit of $D_1 = \Delta_1$ and Ω is a one-cycle whose support doesn't contain the G-orbit of D_1. Since $\mathrm{Pic}^G(X) = \mathbb{Z}$ and is generated by the anti-canonical divisor (Proposition 18) there exists $\xi \in \mathbb{Z}_{>0}$ such that $\Delta_1 + \ldots + \Delta_k \in |-\xi K_X|^G$. Intersecting λD with the anti-canonical class gives

$$4\lambda = \lambda D \cdot (-K_X) \geqslant \lambda d_1 \Big((\Delta_1 + \ldots + \Delta_k) \cdot (-K_X) \Big) > 4\xi.$$

Therefore $\xi = 1$, but this implies that $|-K_X|^G$ is non-empty — contradicting our assumptions.

Hence $\mathrm{LCS}\left(X, \lambda D\right)$ is zero-dimensional and by Corollary 93 consists of at most five points.

Suppose $\mathrm{LCS}\left(X, \lambda D\right)$ consists of precisely

(i) one point. Then by projective duality, there exist a G-invariant hyperplane section of X. This belongs to the anti-canonical linear system — a contradiction.

(ii) two points. Then by duality, there exist a product of hyperplane sections on X that is G-invariant. This belongs to the bi-anti-canonical linear system. However, $|-2K_X|^G$ does not contain any products of hyperplanes for $G = \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ or $\mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}$ — a contradiction.

(iii) three points. If one or more of these points is fixed under the action of G then we must have a G-invariant hyperplane or product of hyperplanes, arguing as before. Thus, the group acts without fixing any points of $\mathrm{LCS}\left(X, \lambda D\right)$. However, for either group there are no orbits of length three. Indeed, for $G = \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ we may see this using direct calculation. For $G = \mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}$, the length of the orbit does not divide the order of G — contradicting the Orbit-Stabiliser theorem.

(iv) four points. Again, by previous arguments, the group must have an orbit of four points. However, by direct calculation this is impossible.

(v) five points. If one or more of these points is fixed under the action of G then we must have a G-invariant hyperplane or product of hyperplanes, arguing as before. Thus, the group acts without fixing any points of $\mathrm{LCS}\left(X, \lambda D\right)$. However, for either group there are no orbits of length five. Indeed, for $G = \mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}$ we may see this using direct calculation. For $G = \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$, the length of the orbit does not divide $|G|$.

$\square$

Theorem 173. *Let X be a smooth minimal del Pezzo G-surface of degree four with the prescribed automorphism group G, then*

$$\operatorname{lct}\big(X, \operatorname{Aut}(\mathrm{X})\big) = \begin{cases} 1 & \textit{if} \operatorname{Aut}(\mathrm{X}) = \mathbb{Z}_2^4, \\ 1 & \textit{if} \operatorname{Aut}(\mathrm{X}) = \mathbb{Z}_2^4 \rtimes \mathbb{Z}_2, \\ 1 & \textit{if} \operatorname{Aut}(\mathrm{X}) = \mathbb{Z}_2^4 \rtimes \mathbb{Z}_4, \\ 2 & \textit{if} \operatorname{Aut}(\mathrm{X}) = \mathbb{Z}_2^4 \rtimes \mathbb{S}_3, \\ 2 & \textit{if} \operatorname{Aut}(\mathrm{X}) = \mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}. \end{cases}$$

6.4.3 Results for individual automorphism groups

Let X be a smooth minimal del Pezzo G-surface of degree four such that $G = \operatorname{Aut}(\mathrm{X})$ and $x_0, \ldots, x_4$ be homogeneous coordinates on $\mathbb{P}^4$. Denote the automorphism $\varphi : X \longrightarrow X$ mapping

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto \big(\varphi(x_0) : \varphi(x_1) : \varphi(x_2) : \varphi(x_3) : \varphi(x_4)\big)$$

by

$$\big[\varphi(x_0), \varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)\big],$$

and let $\epsilon_k = e^{\frac{2\pi i}{k}}$ be the k^{th} primitive root of unity. All notations are described in detail in Chapter 5.

6.4.3.1 $\operatorname{Aut}(\mathrm{X}) = \mathbb{Z}_2^4$

Lemma 174.

$$\operatorname{lct}\big(X, \mathbb{Z}_2^4\big) = \operatorname{lct}_1\big(X, \mathbb{Z}_2^4\big) = 1.$$

Equation of surface and group action

Equation of X:

$$\sum_{i=0}^{4} x_i^2 = \sum_{i=0}^{4} \alpha_i x_i^2 = 0,$$

with $\alpha_i \neq \alpha_j$ for $i \neq j$.

Generators of $\mathrm{Aut}(X)$:

$$s_1 = [x_0, -x_1, x_2, x_3, x_4], \dots, s_4 = [x_0, x_1, x_2, x_3, -x_4].$$

Action of G on $|-K_X|$ The action of $\mathbb{Z}_2^4$ on $H^0\big(X, \mathcal{O}_X(-K_X)\big)$ implies that the only curves $C_i \in |-K_X|^G$ are the hyperplane sections of X, $C_i = \{x_i = 0\}|_X$ for $0 \leqslant i \leqslant 4$.

Singularities of G-invariant curves in $|-K_X|$

Claim 175.

$$\mathrm{lct}_1\big(X, \mathbb{Z}_2^4\big) = 1.$$

Proof. The curves $C_0, \dots, C_4 \in |-K_X|^G$ are smooth curves on X. □

6.4.3.2 Aut(X) = $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$

Lemma 176.

$$\mathrm{lct}\big(X, \mathbb{Z}_2^4 \rtimes \mathbb{Z}_2\big) = 1.$$

Equation of surface and group action

Equation of X:

$$\sum_{i=0}^{4} x_i^2 = x_0^2 + a x_1^2 - x_2^2 - a x_3^2 = 0,$$

with $a \neq \pm 1, 0, i, \pm\frac{1}{i\sqrt{3}}, -2 \pm \sqrt{5}$.

Generators of $\mathrm{Aut}(X)$:

$$s_i = (x_i \mapsto -x_i), g = [x_2, x_3, x_0, x_1, x_4],$$

for $i \in \{1, \dots 4\}$.

Proof of Lemma 176. By Lemma 170, it is enough to exhibit a curve $C \in |-K_X|^G$. Let $C = \{x_4 = 0\}|_X$, then $C \in |-K_X|^G$. □

6.4.3.3 Aut(X) = $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$

Lemma 177.

$$\mathrm{lct}\big(X, \mathbb{Z}_2^4 \rtimes \mathbb{Z}_4\big) = \mathrm{lct}_1\big(X, \mathbb{Z}_2^4 \rtimes \mathbb{Z}_4\big) = 1.$$

Equation of surface and group action

Equation of X:

$$\sum_{i=0}^{4} x_i^2 = x_0^2 + i x_1^2 - x_2^2 - i x_3^2 = 0.$$

Generators of Aut(X):

$$s_i = (x_i \mapsto -x_i), g = [x_1, x_2, x_3, x_0, x_4],$$

for $i \in \{1, \dots 4\}$.

Proof of Lemma 177. By Lemma 170, it is enough to exhibit a curve $C \in |-K_X|^G$. Let $C = \{x_4 = 0\}|_X$, then $C \in |-K_X|^G$. □

6.4.3.4 Aut(X) = $\mathbb{Z}_2^4 \rtimes \mathbb{S}_3$

Lemma 178.

$$\mathrm{lct}\big(X, \mathbb{Z}_2^4 \rtimes \mathbb{S}_3\big) = \mathrm{lct}_2\big(X, \mathbb{Z}_2^4 \rtimes \mathbb{S}_3\big) = 2.$$

Equation of surface and group action

Equation of X:

$$x_0^2 + \epsilon_3 x_1 + \epsilon_3^2 x_2 + x_3^2 = x_0^2 + \epsilon_3^2 x_1^2 + \epsilon_3 x_2^2 + x_4^2 = 0.$$

Generators of $\mathrm{Aut}(X)$:

$$s_i = (x_i \mapsto -x_i), g_1 = [x_0, x_2, x_1, x_4, x_3], g_2 = [x_1, x_2, x_0, \epsilon_3 x_3, \epsilon_3^2 x_4],$$

for $i \in \{1, \ldots 4\}$.

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. Since the only members of $|-K_X|^{\mathbb{Z}_2^4}$ are the curves $C_i = \{x_i = 0\}|_X$ for $i = 0, \ldots, 4$ and these are not invariant under the action of G, $|-K_X|^G$ is empty. □

Action of G on $|-2K_X|$

Claim 179. *The only members of $|-2K_X|^{\mathbb{Z}_2^4}$ are the curves*

$$C = \left\{\lambda_0 x_0^2 + \cdots + \lambda_4 x_4^2 = 0 \,\middle|\, (\lambda_0 : \ldots : \lambda_4) \in \mathbb{P}^4\right\}|_X \quad \textit{and} \quad D = \left\{x_i x_j = 0 \,\middle|\, i \neq j\right\}|_X.$$

Proof. Obvious from the action of $s_1, \ldots, s_4$. □

Examining the action of the generator g_2 on we see that the only curve in $|-2K_X|$ invariant under the action of the whole group G is the curve $C = \{x_0^2 + x_1^2 + x_2^2 = 0\}|_X$.

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2\left(X, \mathbb{Z}_2^4 \rtimes \mathbb{S}_3\right) = 2.$$

Proof. It can be easily checked that $C \in |-2K_X|^G$ is non-singular. □

6.4.3.5 Aut(X) = $\mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}$

Lemma 180.

$$\mathrm{lct}\big(X, \mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}\big) = \mathrm{lct}_2\big(X, \mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}\big) = 2.$$

Equation of surface and group action

Equation of X:

$$\sum_{i=0}^{4} \epsilon_5^i x_i^2 = \sum_{i=0}^{4} \epsilon_5^{4-i} x_i^2 = 0.$$

Generators of Aut(X):

$$s_i = (x_i \mapsto -x_i), g_1 = [x_1, x_2, x_3, x_4, x_0], g_2 = [x_4, x_3, x_2, x_1, x_0],$$

for $i \in \{1, \ldots 4\}$.

Action of G on $|-K_X|$

Claim. $|-K_X|^G = \emptyset$.

Proof. As for $G = \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$. □

Action of G on $|-2K_X|$ By Claim 179 and observing the action of the generators g_1 and g_2, we see that the only curve in $|-2K_X|^G$ is $C = \{x_0^2 + \ldots + x_4^2 = 0\}|_X$.

Singularities of G-invariant curves in $|-2K_X|$

Claim.

$$\mathrm{lct}_2\big(X, \mathbb{Z}_2^4 \rtimes \mathbb{S}_3\big) = 2.$$

Proof. It can be easily checked that $C \in |-2K_X|^G$ is non-singular. □

6.5 Degree Five

6.5.1 Background

Let X be a del Pezzo surface of degree five. Then X is isomorphic to a blow-up

$$\pi : X \longrightarrow \mathbb{P}^2$$

of $\mathbb{P}^2$ in four points, $A_1, \dots, A_4$ in general position, that is, no three lie on a line. The surface X has ten (-1)-curves — these are the pull-backs of the four points $A_1, \dots A_4$; $E_1, \dots, E_4$ and for $1 \leqslant i, j \leqslant 4; i \neq j$ the $\binom{4}{2} = 6$ strict transforms of the lines L_{ij} through pairs of points A_i and A_j we denote by D_{ij} as in Figure 6.6 (see Section 5.1.2). Since we are free to map any four points to any other four under a projective map (all quadrilaterals are similar), it follows that there is only one isomorphism class for X. Thus from Proposition 79 we see that

$$\mathrm{Aut(X)} \cong \mathbb{S}_5.$$

To describe this action, observe that there are five sets of four skew exceptional divisors on X. These are:

- $F_1 = \{E_1, D_{23}, D_{24}, D_{34}\}$;
- $F_2 = \{E_2, D_{13}, D_{14}, D_{34}\}$;
- $F_3 = \{E_3, D_{12}, D_{14}, D_{24}\}$;
- $F_4 = \{E_4, D_{12}, D_{13}, D_{23}\}$;
- $F_5 = \{E_1, E_2, E_3, E_4\}$.

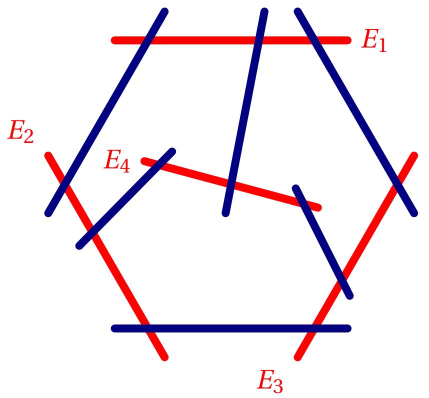


Figure 6.6: The ten (-1)-curves on a del Pezzo surface of degree five.

The action of $\mathbb{S}_5$ on X can be visualised as an action of $\mathbb{S}_4$ that leaves the set F_5 invariant and permutes the sets F_1 to F_4 composed with an element that transposes the sets F_4 and F_5.

6.5.2 General results

Let H be a finite subgroup of Aut(X) and let Z be the divisor formed from the sum of all the exceptional curves on X. Then

$$Z = \sum_{k=1}^{4} E_k + \sum_{i,j=1; i\neq j}^{4} D_{ij},$$

and clearly this is a member of $|-2K_X|^H$, from which follows this Lemma.

Lemma 181. *The pluri-anti-canonical linear system $|-2K_X|$ contains H-invariant members, that is, $|-2K_X|^H \neq \emptyset$.*

From [Che08] (Theorem 33),

$$\mathrm{lct}\big(X, I\big) = \frac{1}{2},$$

where I is the trivial group.

It follows from this and Lemma 181 above that:

Corollary 182.

$$\frac{1}{2} \leqslant \mathrm{lct}\big(X, H\big) \leqslant 2.$$

Dolgachev and Iskovskikh in [DI10] show that the only subgroups, $H \leqslant \mathbb{S}_5$ such that the H-surface $\big(X, H\big)$ is minimal are $\mathbb{S}_5, \mathbb{A}_5, \mathbb{Z}_5 \rtimes \mathbb{Z}_4, \mathbb{Z}_5 \rtimes \mathbb{Z}_2 \cong \mathbb{D}_{10}$, or $\mathbb{Z}_5$. Cheltsov in [Che08] calculates the global H-invariant log canonical thresholds for $\big(X, \mathbb{S}_5\big)$, $\big(X, \mathbb{A}_5\big)$ and $\big(X, \mathbb{Z}_5\big)$ which we present below (cf. Proposition 34). To answer Question C of Section 3.2, it remains to decide on the (weak-)exceptionality of the two H-surfaces $\big(X, \mathbb{Z}_5 \rtimes \mathbb{Z}_2 \cong \mathbb{D}_{10}\big)$ and $\big(X, \mathbb{Z}_5 \rtimes \mathbb{Z}_4\big)$.

Theorem 183. *Let X be a smooth minimal del Pezzo G-surface of degree five with the pre-*

scribed automorphism group G,

$$\mathrm{lct}\big(X,G\big) = \begin{cases} 2 & \textit{if } G = \mathrm{Aut(X)} = \mathbb{S}_5, \\ 2 & \textit{if } G = \mathbb{A}_5, \\ 1 & \textit{if } G = \mathbb{Z}_5 \rtimes \mathbb{Z}_4, \\ \frac{4}{5} & \textit{if } G = \mathbb{Z}_5 \rtimes \mathbb{Z}_2 \cong \mathbb{D}_{10}, \\ \frac{4}{5} & \textit{if } G = \mathbb{Z}_5, \\ \frac{1}{2} & \textit{if } G = I. \end{cases}$$

6.5.3 Results for individual automorphism groups

Let X be a smooth minimal del Pezzo G-surface of degree five such that $G = \mathrm{Aut(X)}$.

6.5.3.1 $\mathrm{Aut(X)} = \mathbb{Z}_5$

Lemma 184 ([Che08, Lemma 5.8])**.**

$$\mathrm{lct}\big(X,\mathbb{Z}_5\big) = \frac{4}{5}.$$

Proof. It is known that the action of $\mathbb{Z}_5 = G$ on X has precisely two fixed points, O_1 and O_2 ([RS02]). There exist five conics on X passing through each point O_1 and O_2. Let the five passing though O_1 be $C_1,\ldots,C_5$, then

$$\sum_{i=1}^{5} C_i \in |-2K_X|^G.$$

The log canonical threshold of this curve is $\frac{2}{5}$, and hence $\mathrm{lct}\big(X,G\big) \leqslant \frac{4}{5}$.

For a contradiction, suppose that $\mathrm{lct}\big(X,G\big) < \frac{4}{5}$.

Then there exists an effective G-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $\big(X,\lambda D\big)$ is not log canonical for some $\lambda \in \mathbb{Q}$ such that $\mathrm{lct}\big(X,G\big) < \lambda < \frac{4}{5}$.

By Lemmata 93 and 96 $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ is zero-dimensional and consists of at most one point, P. Thus $P = O_1$ or O_2 and without loss of generality we may assume $P = O_1$. We may also assume that the support of D doesn't contain the conics $C_1,\dots,C_5$ by Convexity (Lemma 5). Intersecting C_1 and D yields

$$2 = C_1 \cdot D \geqslant \mathrm{mult}_P D. \tag{6.8}$$

Let $\pi : U \longrightarrow X$ be the blowup of X at the point $P = O_1$ with exceptional divisor $E = \pi^{-1}(P)$ and let D_U be the strict transform of D on U. Then, by Remark 11 , there exists a point $Q \in E$ such that

$$\mathrm{mult}_Q D_U \geqslant \frac{2}{\lambda} - \mathrm{mult}_P D > \frac{5}{2} - \mathrm{mult}_P D.$$

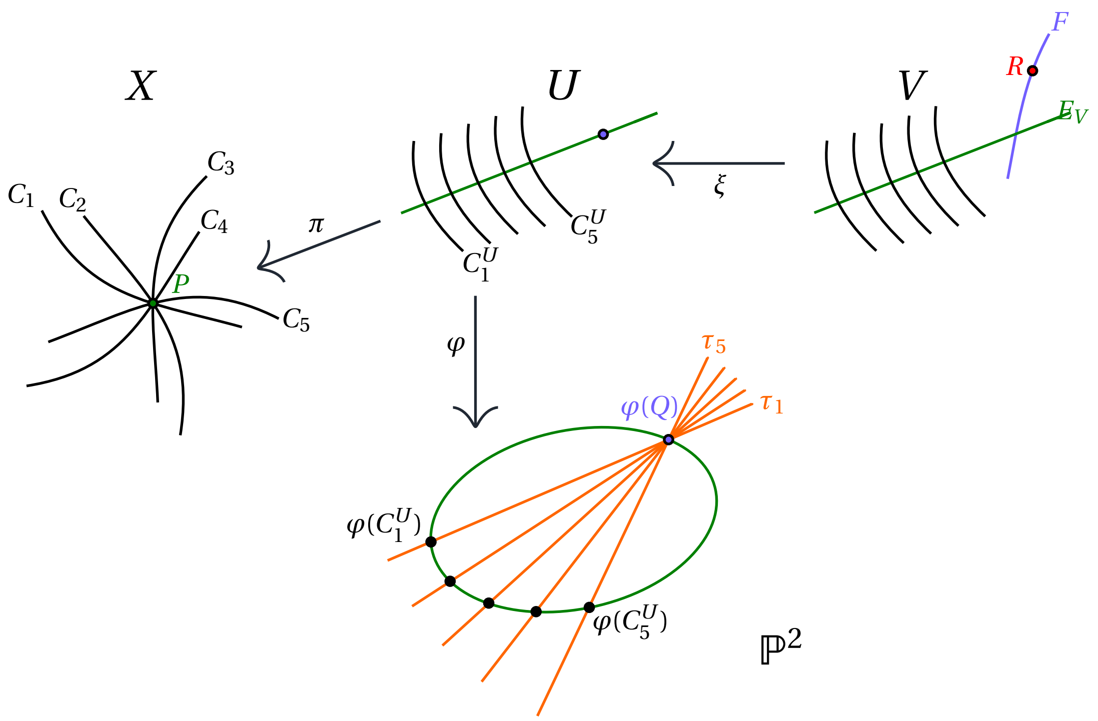


Figure 6.7: $(S_5, \mathbb{Z}_5)$ — the maps $\pi : U \longrightarrow X$, $\varphi : U \longrightarrow \mathbb{P}^2$ and $\xi : V \longrightarrow U$.

The point Q must be G-invariant, since otherwise

$$\mathrm{mult}_P D = E \cdot D_U \geqslant 5\mathrm{mult}_Q D_U > 5(\frac{5}{2} - \mathrm{mult}_P D),$$

which implies that $\mathrm{mult}_P D > 2$, contradicting inequality (6.8).

Write C_i^U for the strict transforms of the conics C_i on U, then $Q \notin \cup_{i=1}^5 C_i^U$ and there exists a birational morphism $\varphi : U \longrightarrow \mathbb{P}^2$ that contracts the C_i^U. Under φ, the π-exceptional divisor E is a conic on $\mathbb{P}^2$ containing the points $\varphi(C_i^U)$. For $i \in \{1,\dots,5\}$, let T_i be the strict transforms of the lines τ_i passing through $\varphi(Q)$ and $\varphi(C_i^U)$ — see Figure 6.7.

Observe that the log pair $\left(X, \frac{\lambda}{3}\sum_{i=1}^{5}\pi(T_i)\right)$ has log terminal singularities,

$$\sum_{i=1}^{5}\pi(T_i) \sim_{\mathbb{Q}} 3D,$$

and by Convexity we may assume that $\mathrm{Supp}(D_U) \not\subseteq T_i$. Intersecting T_1 with D_U gives

$$3 - \mathrm{mult}_P D \geqslant T_1 \cdot D_U \geqslant \mathrm{mult}_Q D_U,$$

that is,

$$\mathrm{mult}_P D + \mathrm{mult}_Q D_U \geqslant 3. \tag{6.9}$$

Let $\xi : V \longrightarrow U$ be the blowup of the point $Q \in E$ with exceptional divisor F and let E_V, D_V be the strict transforms of E, D_U on V, respectively. Then we have

$$K_V + \lambda D_V \sim (\pi\circ\xi)^*\left(K_X + \lambda D\right) + (1 - \lambda\mathrm{mult}_P D)E_V + \left(2 - \lambda\mathrm{mult}_Q D_U - \lambda\mathrm{mult}_P D\right)F$$

hence the log pair

$$\left(V, \lambda D_V + \left(\lambda\mathrm{mult}_P D - 1\right)E_V + \left(\lambda\mathrm{mult}_Q D_U + \lambda\mathrm{mult}_P D - 2\right)F\right) \tag{6.10}$$

is not log terminal at some point $R \in F$.

The point $R \notin E_V$. Indeed, suppose the contrary and let L_V be the strict transform of the line L on $\mathbb{P}^2$ that is tangent to the conic $\varphi(E)$ at the point $\varphi(Q)$. Then $R \in L_V$ and intersecting L_V and D_V gives the contradiction

$$5 - 2\mathrm{mult}_P D - \mathrm{mult}_Q D_U = L_V \cdot D_V \geqslant \mathrm{mult}_R D_V > 5 - 2\mathrm{mult}_P D - \mathrm{mult}_Q D_U.$$

Thus the log pair

$$\left(V, \lambda D_V + (\lambda\mathrm{mult}_Q D_U + \lambda\mathrm{mult}_P D - 2)F\right)$$

is not log terminal at R.

Observe that the divisor $\lambda D + \big(\lambda \mathrm{mult}_Q D_U + \lambda \mathrm{mult}_P D - 2\big)F$ is effective, hence

$$\mathrm{mult}_R D_V > \frac{3}{\lambda} - \mathrm{mult}_Q D_U - \mathrm{mult}_P D > \frac{15}{4} - \mathrm{mult}_Q D_U - \mathrm{mult}_P D.$$

Writing T_i^V for the strict transforms of the T_i on V for $i \in \{1,\ldots,5\}$, suppose that $R \in T_k^V$ for some $1 \leqslant k \leqslant 5$. Then we have that

$$3 - \mathrm{mult}_Q D_U - \mathrm{mult}_P D = T_k^V \cdot D_V \geqslant \mathrm{mult}_P D_V > \frac{15}{4} - \mathrm{mult}_Q D_U - \mathrm{mult}_P D$$

— a contradiction. Hence $R \notin \cup_{i=1}^5 T_i^V$.

Let M be an irreducible curve on V such that $R \in M$. Then $\varphi \circ \xi(M)$ is a line on $\mathbb{P}^2$ passing through $\varphi(Q)$, which implies that $\pi \circ \xi(M)$ has an ordinary double point at $P \in X$. However, since we may assume that $M \nsubseteq \mathrm{Supp}(D)$ and $\pi \circ \xi(M) \sim_{\mathbb{Q}} -K_X$ as $R \notin \cup_{i=1}^5 T_i^V$ we have that

$$5 - 2\mathrm{mult}_P D - \mathrm{mult}_Q D_U \geqslant M \cdot D_V \geqslant \mathrm{mult}_R D_V > \frac{15}{4} - \mathrm{mult}_P D - \mathrm{mult}_Q D_U$$

that is

$$\mathrm{mult}_P D \leqslant \frac{5}{4}$$

— a contradiction, since $\mathrm{mult}_P D > \frac{5}{4}$. $\square$

Remark 185. The curve $\sum_{i=1}^5 C_i \in |-2K_X|^{\mathbb{Z}_5}$ is the only example of a wild tiger, Δ on a surface S, we found in $|-mK_S|^{\mathrm{Aut(S)}}$ such that $\mathrm{lct}\big(S, \mathrm{Aut(S)}\big) \leqslant \mathrm{m} - 1$.

6.5.3.2 Aut(X) $= \mathbb{Z}_5 \rtimes \mathbb{Z}_2 \cong \mathbb{D}_{10}$

Lemma 186.

$$\mathrm{lct}\big(X, \mathbb{D}_{10}\big) = \frac{4}{5}.$$

Proof. Since $\mathbb{Z}_5$ is a subgroup, it follows from Lemma 184 that $\mathrm{lct}\big(X,G\big) \geqslant \frac{4}{5}$. Suppose that $\mathrm{lct}\big(X,G\big) < \frac{4}{5}$, then there exists an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair $\big(X, \lambda D\big)$ is not log canonical for $\lambda \in \mathbb{Q}$ with $\mathrm{lct}\big(X,G\big) < \lambda < \frac{4}{5}$.

By Lemmata 93 and 96 $\mathrm{LCS}\big(\mathrm{X},\lambda\mathrm{D}\big)$ is zero-dimensional and consists of at most one point, P. This point P must be unique in its G-orbit and hence is fixed. Passing through it are five conics $C_1,\dots,C_5$ and their sum $C=\sum_{i=0}^{5}C_i$ is a G-invariant member of $|-2K_X|$. By Convexity (Corollary 6), we may assume that $C_i \nsubseteq \mathrm{Supp}(D)$ for $1\leqslant i\leqslant 5$ and hence

$$2 = C_1\cdot D \geqslant \mathrm{mult}_P D.$$

We may now follow the construction of the proof of Lemma 184 from inequality (6.8) reaching a contradiction as before. □

Lemma 187.

$$\mathrm{lct}\big(X,G\big)\leqslant 1.$$

Proof. We prove that there exists a G-invariant curve in the anti-canonical linear system $|-K_X|$. First let us let us examine the action of $\mathbb{S}_5$ on $H^0\big(X,\mathcal{O}_X(-K_X)\big)$. The symmetric group $\mathbb{S}_5$ has seven conjugacy classes:

$$\left.\begin{array}{l}\Sigma_1\\ \Sigma_2\\ \Sigma_3\\ \Sigma_4\\ \Sigma_5\\ \Sigma_6\\ \Sigma_7\end{array}\right\}\text{ containing the }\left\{\begin{array}{l}\text{identity;}\\ \text{ten two-cycles;}\\ \text{twenty three-cycles;}\\ \text{thirty four-cycles;}\\ \text{twenty-four five-cycles;}\\ \text{fifteen even elements of order two;}\\ \text{twenty elements of order six.}\end{array}\right.$$

The character table of $\mathbb{S}_5$ is shown in Table 6.1.

It is known that the $\mathbb{S}_5$-action on $H^0\big(X,\mathcal{O}_X(-K_X)\big)\cong\mathbb{C}^6$ is the unique irreducible six-dimensional action with character vector $\chi_7=(6,0,0,0,1,-2,0)$ ([RS02, Theorem 2], cf. Table 6.1).

class	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7
size	1	10	20	30	24	15	20
order	1	2	3	4	5	2	6
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	1	-1	-1	0	1	1
χ_6	5	-1	-1	1	0	1	-1
χ_7	6	0	0	0	1	-2	0

Table 6.1: Character Table for $\mathbb{S}_5$.

To show that there is a curve in $|-K_X|^G$ we show that the action of $\mathbb{D}_{10}$ on

$$H^0\big(X, \mathcal{O}_X(-K_X)\big) \cong \mathbb{C}^6$$

splits as $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$. The dihedral group

$$\mathbb{D}_{10} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_2 = \left\langle a, x \,\middle|\, a^5 = x^2 = 1, xax^{-1} = a^{-1} \right\rangle$$

has character table given in Table 6.2 (φ is the golden mean $\frac{1+\sqrt{5}}{2}$; $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$) and four conjugacy classes:

$$\left.\begin{array}{c} \Gamma_1 \\[2ex] \Gamma_2 \\[2ex] \Gamma_3 \\[2ex] \Gamma_4 \end{array}\right\} = \left\{\begin{array}{l} \{\text{identity}\}; \\[2ex] \{a, a^{-1}\}; \\[2ex] \{a^2, a^{-2}\}; \\[2ex] \{x, ax, a^2x, a^3x, a^4x\}. \end{array}\right.$$

class	Γ_1	Γ_2	Γ_3	Γ_4
size	1	2	2	5
order	1	5	5	2
γ_1	1	1	1	1
γ_2	1	1	1	-1
γ_3	2	$-\varphi$	$-\bar{\varphi}$	0
γ_4	2	$-\bar{\varphi}$	$-\varphi$	0

Table 6.2: Character Table for $\mathbb{D}_{10}$.

To understand how the action of $\mathbb{D}_{10}$ on $H^0\big(X, \mathcal{O}_X(-K_X)\big) \cong \mathbb{C}^6$ splits we consider how to write the restricted character $\chi_7|_{\mathbb{D}_{10}}$ as a sum of the irreducible characters of D_{10}. Observe then the inclusions of the conjugacy classes of $\mathbb{D}_{10}$ in those of $\mathbb{S}_5$: $\Gamma_1 \subseteq \Sigma_1$, $\Gamma_2 \subseteq \Sigma_5$, $\Gamma_3 \subseteq \Sigma_5$, $\Gamma_4 \subseteq \Sigma_6$. It follows that

$$\chi_7|_{\mathbb{D}_{10}} = 2\gamma_2 + \gamma_3 + \gamma_4$$

and hence the action splits as advertised as $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$ and so we must have G-invariant curves in $|-K_X|$. □

6.5.3.3 Aut(X) = $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$

Lemma 188.

$$\operatorname{lct}\big(X, \mathbb{Z}_5 \rtimes \mathbb{Z}_4\big) = 1.$$

Proof. The group $\mathbb{Z}_5 \rtimes \mathbb{Z}_4 = G$ acts without fixed points, thus it follows from Lemma 28 that $\operatorname{lct}\big(X, G\big) \geqslant 1$. Furthermore from Lemma 189, $\operatorname{lct}\big(X, G\big) \leqslant 1$ — hence result. □

Lemma 189.

$$\operatorname{lct}\big(X, G\big) \leqslant 1.$$

Proof. Similarly to the previous Lemma, we show that the action of $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ on

$$H^0\big(X, \mathcal{O}_X(-K_X)\big) \cong \mathbb{C}^6$$

splits as $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^4$ by examining the restricted character $\chi_7|_{\mathbb{Z}_5 \rtimes \mathbb{Z}_4}$.

The group

$$\mathbb{Z}_5 \rtimes \mathbb{Z}_4 = \left\langle a, x \;\middle|\; a^5 = x^4 = 1, xax^{-1} = a^2 \right\rangle$$

has character table as shown in Table 6.3 and five conjugacy classes:

$$\left.\begin{array}{l}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\\ \Psi_5\end{array}\right\} = \left\{\begin{array}{l}\{\,\text{identity}\};\\ \{a, a^2, a^3, a^4\};\\ \{x, ax, a^2x, a^3x, a^4x\};\\ \{x^2, ax^2, a^2x^2, a^3x^2, a^4x^2y\};\\ \{x^3, ax^3, a^2x^3, a^3x^3, a^4x^3y\}.\end{array}\right.$$

class	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5
size	1	4	5	5	5
order	1	5	2	4	4
ψ_1	1	1	1	1	1
ψ_2	1	1	1	-1	-1
ψ_3	1	1	1	i	$-i$
ψ_4	1	1	1	$-i$	i
ψ_5	4	-1	0	0	0

Table 6.3: Character Table for $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

We have the inclusions of the conjugacy classes of $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ in those of $\mathbb{S}_5$: $\Psi_1 \subseteq \Sigma_1$, $\Psi_2 \subseteq \Sigma_5$, $\Psi_3 \subseteq \Sigma_6$, $\Psi_4, \Psi_5 \subseteq \Sigma_4$. It follows that

$$\chi_7|_{\mathbb{Z}_5 \rtimes \mathbb{Z}_4} = \psi_1 + \psi_2 + \psi_5,$$

and hence the action splits as $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^4$ and it follows that we have G-invariant curves in $|-K_X|$. □

6.5.3.4 Aut(X) = $\mathbb{A}_5$

Lemma 190 ([Che08, Lemma 5.7])**.**

$$\operatorname{lct}\big(X, \mathbb{A}_5\big) = 2.$$

Proof. The result follows from Lemma 97, the fact that $\mathbb{A}_5$ has no orbits of length less than or equal to five and that the $\mathbb{A}_5$-invariant anti-canonical linear system $|-K_X|^{\mathbb{A}_5}$ is empty.

Indeed, suppose that $\mathbb{A}_5$ has an orbit of length n where $1 \leqslant n \leqslant 5$. Then, by the Orbit-Stabiliser Theorem, the stabiliser H must have order $60, 30, 20, 15, 24$ or 10. As $\mathbb{A}_5$ has only subgroups of orders $1, 2, 3, 4, 5, 6, 10, 12$ or 60, the only possibility is that $H \cong \mathbb{A}_5$. However, $\mathbb{A}_5$ doesn't have a faithful two-dimensional representation — contradicting the fact that the stabiliser of a point acts faithfully on its tangent space.

There are no $\mathbb{A}_5$-invariant divisors in the anti-canonical linear system: $\mathbb{A}_5$ has five conjugacy classes — the even conjugacy classes of $\mathbb{S}_5$; $\Sigma_1, \Sigma_3, \Sigma_6$ and two conjugacy classes Σ_{5a}, Σ_{5b} whose union is the conjugacy class Σ_5 of $\mathbb{S}_5$. The character table of $\mathbb{A}_5$ is shown in Table 6.4, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$.

class	Σ_1	Σ_3	Σ_{5a}	Σ_{5b}	Σ_6
size	1	20	12	12	15
order	1	3	5	5	2
ξ_1	1	1	1	1	1
ξ_2	3	0	φ	$\bar{\varphi}$	-1
ξ_3	4	1	-1	-1	0
ξ_4	3	0	$\bar{\varphi}$	φ	0
ξ_5	5	-1	0	0	1

Table 6.4: Character Table for $\mathbb{A}_5$.

It is known that the $\mathbb{S}_5$-action on $H^0\big(X, \mathcal{O}_X(-K_X)\big) \cong \mathbb{C}^6$ is the unique irreducible six-dimensional action with character vector $\chi_7 = (6, 0, 0, 0, 1, -2, 0)$ ([RS02, Theorem 2], cf. Table 6.1). The $\mathbb{S}_5$-character χ_7 restricted to $\mathbb{A}_5$ can be written as

$$\chi_7|_{\mathbb{A}_5} = \xi_2 + \xi_4,$$

which shows that the action of $\mathbb{A}_5$ on $H^0\big(X, \mathcal{O}_X(-K_X)\big) \cong \mathbb{C}^6$ splits as $\mathbb{C}^3 \oplus \mathbb{C}^3$ and hence

$$|-K_X|^{\mathbb{A}_5} = \emptyset. \qquad \square$$

6.5.3.5 Aut(X) = $\mathbb{S}_5$

Lemma 191 ([Che08, Example 5.5])**.**

$$\operatorname{lct}\big(X, \mathbb{S}_5\big) = 2.$$

Proof. This is an easy application of Lemma 97 and Lemma 181. Using the facts that $\mathbb{S}_5$ has no orbits of length less than or equal to six and $|-K_X|^{\mathbb{S}_5}$ is empty.

These are both easy to see: For the first, suppose that $\mathbb{S}_5$ has an orbit of length n where $1 \leqslant n \leqslant 6$. Then, by the Orbit-Stabiliser Theorem, the stabiliser H must have order $120, 60, 40, 30, 24$ or 20. As $\mathbb{S}_5$ has no subgroups of order 30 or 40, the only possibilities for H are $\mathbb{S}_5, \mathbb{A}_5, \mathbb{S}_4$ or the subgroup of order 20 generated by a five-cycle and a four-cycle. However, none of these subgroups has a faithful two-dimensional representation — contradicting the fact that the stabiliser of a point acts faithfully on its tangent space.

For the second, it is known that the $\mathbb{S}_5$-action on $H^0\big(X, \mathcal{O}_X(-K_X)\big) \cong \mathbb{C}^6$ is the unique irreducible six-dimensional action with character vector $\chi_7 = (6,0,0,0,1,-2,0)$ ([RS02, Theorem 2], cf. Table 6.1). $\square$

6.6 Degree Six

6.6.1 Background

Let X be a del Pezzo surface of degree six. We may describe X equivalently as:

- the blow-up of $\mathbb{P}^2$ in three non-collinear points P_1, P_2, P_3. Since any three points in $\mathbb{P}^2$ are equivalent under linear automorphisms of $\mathbb{P}^2$ there is only one isomorphism class. From Section 5.1.2 we see that X has six (-1)-curves — three exceptional divisors E_1, E_2, E_3 over the blown-up points P_1, P_2, P_3, respectively on $\mathbb{P}^2$ and the strict transform L_{ij} of the three lines though any two of the points P_i, P_j for $1 \leqslant i, j \leqslant 3$; $i \neq j$. These lines are arranged in a hexagon on X (as in Figure 6.8);

- the image in $\mathbb{P}^6$ of the rational map $\varphi : \mathbb{P}^2 \longrightarrow \mathbb{P}^6$ which maps

$$(x:y:z) \mapsto (x^2y : x^2z : xy^2 : xyz : xz^2 : y^2z : yz^2),$$

  where x, y, z are homogeneous coordinates on $\mathbb{P}^2$. This map is given by the linear system of cubics passing through the points $A_1 = (1:0:0)$, $A_2 = (0:1:0)$ and $A_3 = (0:0:1)$;

- the set

$$\left\{((x_1 : y_1 : z_1), (x_2 : y_2 : z_2)) \middle| x_1x_2 = y_1y_2 = z_1z_2\right\} \subset \mathbb{P}^2 \times \mathbb{P}^2,$$

  where for $i = 1, 2$; x_i, y_i, z_i are homogeneous coordinates on each factor of $\mathbb{P}^2$. The projection on to one factor contracts E_1, E_2, E_3 and the projection on to the other contracts the L_{12}, L_{13}, L_{23};

- the set

$$\{X_0X_1X_2 = Y_0Y_1Y_2\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

  where for $i = 0, 1, 2$; X_i, Y_i are homogeneous coordinates on each factor of $\mathbb{P}^1$. In fact, the map $f : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$f_x\big((x_0 : y_0 : z_0), (x_1 : y_1 : z_1)\big) = (y_i : z_i) = (X_0 : Y_0),$$
$$f_y\big((x_0 : y_0 : z_0), (x_1 : y_1 : z_1)\big) = (x_i : z_i) = (X_1 : Y_1),$$
$$f_z\big((x_0 : y_0 : z_0), (x_1 : y_1 : z_1)\big) = (x_i : y_i) = (X_2 : Y_2),$$

map isomorphically $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ to $\{X_0X_1X_2 = Y_0Y_1Y_2\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

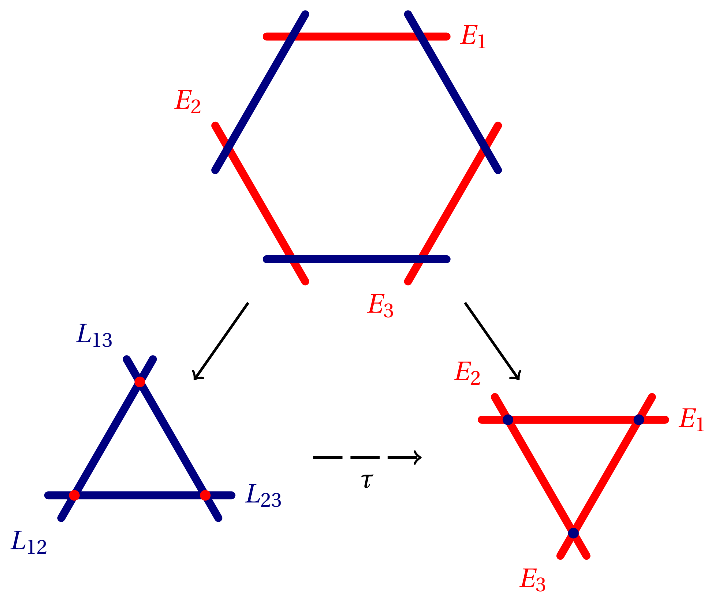


Figure 6.8: The standard Cremona quadratic transformation, τ.

Remark 192. The birational morphism of $\mathbb{P}^2$ induced by the projection of $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ to either factor is the famous standard Cremona quadratic transformation of $\mathbb{P}^2$ (Figure 6.8)

$$\tau : (x : y : z) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right).$$

Write $G = \mathrm{Aut(X)}$, then by Lemma 78 Aut(X) has infinite order. In fact, X is toric with a fan given by the dual of the hexagon of lines E_1, E_2, E_3 and L_{12}, L_{13}, L_{23} (see, for example, [Blu10]). In addition to the torus action on X there is the action of $\mathbb{Z}_2$ and $\mathbb{S}_3$. $\mathbb{Z}_2$ acts by changing the factors $\mathbb{P}^2$ of $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{S}_3$ arises from the diagonal action on the coordinates $x_0, y_0, z_0; x_1, y_1, z_1$. The actions of S_3 and $\mathbb{Z}_2$ commute and we have the split short exact sequence

$$1 \longrightarrow T \longrightarrow \mathrm{Aut(X)} \longrightarrow \mathbb{Z}_2 \times \mathbb{S}_3 \longrightarrow 1,$$

where T is the torus acting on X. Another way to see this is to examine the Weyl group $\mathcal{W}_X$, it turns out ([DI10, Section 6.2]) that $\mathcal{W}_X = \mathbb{Z}_2 \times \mathbb{S}_3$ and the representation $\rho : \mathrm{Aut(X)} \longrightarrow \mathcal{W}_\mathrm{X}$ is surjective with kernel T.

6.6.2 General results

The divisor formed by the sum of all six of the lines is clearly H-invariant for any subgroup H of the full automorphism group, as this is also a member of the first anti-canonical linear system we have that

$$\mathrm{lct}\big(X,H\big) \leqslant 1.$$

Hence there are no exceptional del Pezzo H-surfaces of degree six.

From [Che08] (Proposition 33) we know that

$$\mathrm{lct}\big(X,I\big) = \frac{1}{2},$$

where I is the trivial group.

Thus for any subgroup H of Aut(X) we have:

Lemma 193.

$$\frac{1}{2} \leqslant \mathrm{lct}\big(X,H\big) \leqslant 1.$$

6.6.3 Weakly-exceptional criterion

To answer Question C of Section 3.2 we need to list all minimal pairs $\big(X,G\big)$ such that $\mathrm{lct}\big(X,G\big) = 1$. From [DI10, Theorem 6.3], we know that subgroups of Aut(X) such that the pair $\big(X,G\big)$ is minimal are isomorphic to $H_\bullet\langle\tilde{\tau}\rangle$, where $\tilde{\tau}$ is the lift of the standard quadratic transformation on $\mathbb{P}^2$ (see Remark 192) under the blowup map and H is a imprimitive finite subgroup of $\mathbb{PGL}_3(\mathbb{C})$ (Definition 63). A list of imprimitive subgroups of $\mathbb{PGL}_3(\mathbb{C})$ can be found in Section 6.9.

Theorem 194. *For a smooth del Pezzo G-surface S_6 of degree six such that* $\mathrm{Pic}^{G}(S_6) = \mathbb{Z}$,

$$\mathrm{lct}\big(S_6, G\big) = 1$$

if, and only if, $\big(S_6, G\big)$ *has no G-fixed points.*

Proof. The reverse implication is the content of Lemma 28. Suppose then that $\big(X, G\big)$ is weakly-exceptional and assume for a contradiction that there exists a G-fixed point P on X. There are six (-1)-curves, E_1, E_2, E_3 and L_{12}, L_{13}, L_{23}, arranged in a hexagon on X and we may contract either set of three disjoint (-1)-curves to get a birational morphism $\sigma : X \longrightarrow \mathbb{P}^2$. Writing Ξ_i for the curves contracted under σ, we may assume that $\sigma(P) \notin \Xi_i$ since otherwise the divisor Ξ_i is G-invariant and the rank of the G-Picard group is greater than one. However, this assumption also leads to a contradiction. Let $Q_i = \sigma(\Xi_i)$; Λ_i be the lines on $\mathbb{P}^2$ passing through Q_i and the point $\sigma(P)$; and $\overline{\Lambda_i}$ the strict transforms of the lines Λ_i. Then $D = \sum_{i=1}^{3} (\overline{\Lambda_i} + \Xi_i) \equiv -K_X$ is an effective G-invariant divisor on X such that $\mathrm{lct}\big(X, D\big) = \frac{1}{3}$. □

Example 195. Let $G = \mathbb{Z}_2^2 \rtimes \mathbb{S}_3$, acting on $X = \big\{X_0 Y_0 Z_0 - X_1 Y_1 Z_1 = 0\big\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $G \cong \mathbb{S}_4$, $\big(X, G\big)$ is a del Pezzo G-surface of degree six and as G acts without fixed points ([BT05, Section 1.1]),

$$\mathrm{lct}\big(X, G\big) = 1.$$

Since G also is of the form $H_{\bullet}\langle\widetilde{\tau}\rangle$ with H imprimitive, $Pic^{G}(X) = \mathbb{Z}$ ([DI10, Theorem 6.3]) — that is,

$$\big(X, \mathbb{S}_4\big) \text{ is a minimal weakly-exceptional pair.}$$

6.6.4 Further research

To answer Question C fully, we need to run through all the imprimitive finite subgroup of $\mathbb{PGL}_3(\mathbb{C})$, H and decide which groups $H_{\bullet}\langle\widetilde{\tau}\rangle$ have fixed points. We leave this for further research. Another approach to answer Question C, is to use the following theorem of Cheltsov and Shramov.

Let $N = \mathbb{Z}^n$ be a lattice of rank n, and let $M = \mathrm{Hom}\big(N, \mathbb{Z}\big)$ be the dual lattice. Let V be a toric variety defined by a complete fan $\Sigma \subset N_{\mathbb{R}}$, let

$$\Delta_1 = \{v_1, \ldots, v_m\}$$

be a set of generators of one-dimensional cones of the fan Σ. Put

$$\Delta = \left\{ w \in M \,\middle|\, \langle w, v_i \rangle \geqslant -1 \text{ for all } i = 1, \ldots, m \right\}.$$

Put $T = (\mathbb{C}^*)^n \subset \mathrm{Aut}(V)$. Let $\mathcal{N}$ be the normaliser of T in $\mathrm{Aut}(V)$ and $\mathcal{W} = \mathcal{N}/T$.

Lemma 196 ([CS08, Lemma 6.1])**.** *Let $G \subset \mathcal{W}$ be a subgroup. Suppose that V is $\mathbb{Q}$-factorial. Then*

$$\mathrm{lct}\big(V, G\big) = \frac{1}{1 + \max\left\{\langle w, v \rangle \mid w \in \Delta^G,\ v \in \Delta_1\right\}},$$

where Δ^G is the set of the points in Δ that are fixed by the group G.

6.7 Non-Kähler-Einstein del Pezzo Surfaces

Suppose X is a smooth del Pezzo surface that does not admit a Kähler-Einstein metric, then X is the blowup of $\mathbb{P}^2$ in one or two points by Theorem 53. Necessarily, for any finite group H acting regularly on X, this means that neither of these possibilities can be H-exceptional as by the contra-positive of Theorem 52, $\operatorname{lct}(X, H) \leqslant \frac{2}{3}$.

6.7.1 Degree seven

Let X be a del Pezzo surface of degree seven with $H \leqslant \mathrm{Aut(X)}$ a finite (sub-)group of automorphisms. Recall from Section 5.1.2, that X contains three lines — the exceptional curves E_1 and E_2 over the points of the blow-up of $\mathbb{P}^2$ and the strict transform L of the line on $\mathbb{P}^2$ through these points. Thus we see that (X, H) is always non-minimal; indeed, for any group acting on X, L must be unique in its orbit and thus we may equivariantly blow it down yielding a H-equivariant map to $\mathbb{P}^1 \times \mathbb{P}^1$.

Observe that

$$3L + 2E_1 + 2E_2 \in |-K_X|^H.$$

Hence, $\operatorname{lct}(X, H) \leqslant \frac{1}{3}$ for any $H \leqslant \mathrm{Aut(X)}$ and so there are no exceptional del Pezzo H-surfaces of degree seven. In fact, since $\operatorname{lct}(X, I) = \frac{1}{3}$ we have the following.

Theorem 197.

$$\operatorname{lct}(X, H) = \frac{1}{3},$$

for any group $H \leqslant \mathrm{Aut(X)}$.

6.7.2 The blow-up of $\mathbb{P}^2$ in one point, $\mathbb{F}_1 = \mathbb{P}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right)$

Let $\mathbb{F}_1 = \mathbb{P}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right)$. Then $\mathbb{F}_1$ is clearly non-minimal as it has a unique (-1)-curve. It is easy to see that the automorphisms of $\mathbb{F}_1$ is the group of automorphisms of the projective plane fixing one point. That is, if $P \in \mathbb{P}^2$ is the point of the blow-up then

$$\mathrm{Aut}(\mathbb{F}_1) \cong \left\{ \mathrm{g} \in \mathbb{PGL}_3(\mathbb{C}) \,\middle|\, \mathrm{g(P) = P} \right\}.$$

From Theorem 33 we know that $\operatorname{lct}(\mathbb{F}_1) = \frac{1}{3}$. It follows from this and the contra-positive of Theorem 52 that we have for any finite $H \leqslant \operatorname{Aut}(\mathbb{F}_1)$ the following lemma.

Lemma 198.

$$\frac{1}{3} \leqslant \operatorname{lct}\big(\mathbb{F}_1, H\big) \leqslant \frac{2}{3}.$$

6.8 The Smooth Quadric Cone

Let $X = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then

$$\operatorname{Aut}(\mathrm{X}) \cong \mathbb{PO}_4(\mathbb{C}) \cong \mathbb{PGL}_2(\mathbb{C}) \wr \mathbb{S}_2 \cong \big(\mathbb{PGL}_2(\mathbb{C})\big)^2 \rtimes \mathbb{S}_2,$$

where $\mathbb{PO}_4(\mathbb{C})$ is the projective orthogonal group of dimension four over $\mathbb{C}$ and $\wr$ is the wreath product. As we alluded to above, we can answer at present Question B only for those subgroups of Aut(X) which are of the form $A \times A$ for $A \leqslant \mathbb{PGL}_2(\mathbb{C})$. To answer this question fully we hope to continue as described in Section 6.8.1.

Let us examine finite subgroups of $\mathbb{PGL}_2(\mathbb{C})$. These are well known to be isomorphic to one of the following polyhedral groups:

- a cyclic group, C_n;
- a dihedral group, $\mathbb{D}_{2n}$;
- the tetrahedral group, $T \cong \mathbb{A}_4$ of order 12;
- the octahedron group, $O \cong \mathbb{S}_4$ of order 24;
- the icosahedron group, $I \cong \mathbb{A}_5$ of order 60.

First we recall the answer to Question B for $\mathbb{P}^1$. Shokurov shows in [Sho93] (cf. [MP99b, Example 1.5] and [CS09, Theorem 1.25]) that for a two-dimensional quotient singularity $\big(V \ni P\big) = \big(\mathbb{C}^2 \ni 0\big)/G$, $\big(V \ni P\big)$ is exceptional if and only if the corresponding image in $\mathbb{PGL}_2(\mathbb{C})$, $\pi(G)$, is either dihedral, tetrahedral or icosahedral. If Conjecture 67 is true, then we have all groups such that $\operatorname{lct}\big(\mathbb{P}^1, \pi(G)\big) > 1$. Otherwise suppose that G is not dihedral, tetrahedral or the icosahedron group. Then $\mathrm{lct}\big(\mathbb{P}^1, \pi(G)\big) \leqslant 1$ and so the pair $\big(\mathbb{P}^1, \pi(G)\big)$ is non-exceptional and we are done.

The following result is a generalisation of Lemma 2.30 in [CS08] taking into account an action of a finite group G. In fact, the proof goes through verbatim.

Theorem 199 (cf. [CS08, Lemma 2.30])**.** *Let U and V be smooth Fano varieties such that a finite group G acts on both. Then*

$$\mathrm{lct}\big(U \times V, G_U \times G_V\big) = \min\Big(\mathrm{lct}\big(U, G_U\big), \mathrm{lct}\big(V, G_V\big)\Big),$$

where G_U, G_V are the actions of G on U and V respectively.

Proof. It is clear that $\mathrm{lct}\big(U, G_U\big)$, $\mathrm{lct}\big(V, G_V\big) \geqslant \mathrm{lct}\big(U \times V, G_U \times G_V\big)$. Suppose, to obtain a contradiction, that

$$\mathrm{lct}\big(U \times V, G_U \times G_V\big) < \min\Big(\mathrm{lct}\big(U, G_U\big), \mathrm{lct}\big(V, G_V\big)\Big).$$

Then there exists a $G_U \times G_V$-invariant effective $\mathbb{Q}$-divisor Δ such that $\Delta \sim_{\mathbb{Q}} -K_{U\times V}$ and the log pair $\big(U \times V, \lambda\Delta\big)$ is not log canonical at some point $P \in U \times V$ where

$$\lambda < \min\Big(\mathrm{lct}\big(U, G_U\big), \mathrm{lct}\big(V, G_V\big)\Big).$$

Identify V with the fibre of the projection $U \times V \longrightarrow U$ containing the point P. Suppose that

$$\mathrm{lct}_V\big(U \times V, \Delta\big) \geqslant \mathrm{lct}\big(V, D_V\big),$$

where $D_V \sim \Delta|_V$. Clearly $D_V \sim -K_V$ and is G_V-invariant, hence

$$\mathrm{lct}\big(V, D_V\big) = \mathrm{lct}\big(V, -K_V\big) = \mathrm{lct}\big(V, G_V\big).$$

However we also have that

$$\mathrm{lct}\big(V, G\big) > \lambda \geqslant \mathrm{lct}_V\big(U \times V, \Delta\big),$$

by the definitions of λ and $\mathrm{lct}_V\big(U \times V, \Delta\big)$. Using Hwang's Product Theorem (included below as Proposition 200), we see that for all points $O,Q \in V$ we have that

$$\mathrm{lct}_O\big(U \times V, \Delta\big) = \mathrm{lct}_Q\big(U \times V, \Delta\big).$$

Hence $\big(U \times V, \lambda\Delta\big)$ is not log canonical at all points of $V \subseteq U \times V$, which implies that $V \in \mathbb{LCS}\big(U \times V, \lambda\Delta\big)$.

Identifying U with a general fibre of the projection $U \times V \longrightarrow V$, we see that $D_U \sim_{\mathbb{Q}} -K_U$ is G_U-invariant, where $D_U \sim \Delta|_U$. However, by Lemma 89 (applied $\dim(V)$ times)

$$V \in \mathbb{LCS}\big(U \times V, \lambda\Delta\big) \Longleftrightarrow U \cap V \in \mathbb{LCS}\big(U, \lambda D_U\big)$$

that is to say that the log pair $\big(U, \lambda D_U\big)$ is not log canonical in $U \cap V$ — however $\lambda < \mathrm{lct}\big(U, G_U\big)$, a contradiction. □

Proposition 200 ([Hwa07, Product Theorem]; [CS08, Theorem 2.28])**.** *Let X, Y be varieties with log terminal singularities, let $\varphi : X \longrightarrow Y$ be a surjective morphism such that φ is smooth in a neighbourhood of a fibre F of φ and let Δ be an effective $\mathbb{Q}$-divisor on X. Then either*

$$\mathrm{lct}_F\big(X, \Delta\big) \geqslant \mathrm{lct}\big(F, D\big)$$

where $D \sim \Delta|_F$, or

$$\mathrm{lct}_O\big(X, \Delta\big) = \mathrm{lct}_Q\big(X, \Delta\big)$$

for all points $O, Q \in F$.

Corollary 201. *Let A be a finite subgroup of $\mathbb{PGL}_2(\mathbb{C})$ such that $\mathrm{lct}\big(\mathbb{P}^1 \times \mathbb{P}^1, A \times A\big) > 1$. Then $A \cong \mathbb{A}_5, \mathbb{A}_4$ or $\mathbb{D}_{2n}$ for some $n \in \mathbb{N}$.*

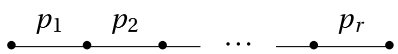
6.8.1 Further research

What other finite subgroups of $\mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1\big)$ remain to be checked? Goursat in his 1889 paper [Gou89] classified finite subgroups of $\mathbb{O}_4(\mathbb{C})$. It turns out that a finite subgroups of $\mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1\big)$ is either of the form $A \times B$ for $A, B \leqslant \mathbb{PGL}_2(\mathbb{C})$ or it is one of the groups in Table 6.5 ([DI10, Theorem 4.9]).

$\frac{1}{60}[I\times I]$	$\cong$	I	$\cong$	$[3,5]^+$	
$\frac{1}{60}[I\times I]$	$\cong$	I	$\cong$	$[3,3,3]^+$	
$\frac{1}{24}[O\times O]$	$\cong$	O	$\cong$	$[3,4]^+$	
$\frac{1}{24}[O\times O]$	$\cong$	O	$\cong$	$[2,3,3]^+$	
$\frac{1}{12}[T\times T]$	$\cong$	T	$\cong$	$[3,3]^+$	
$\frac{1}{2}[O\times O]$	$\cong$	$\big(T\times T\big)\rtimes\mathbb{Z}_2$	$\cong$	$\overline{[3,4,3]^+}$	
$\frac{1}{6}[O\times O]$	$\cong$	$\mathbb{Z}_2^4\rtimes\mathbb{S}_3$	$\cong$	$\overline{[3,3,4]^+}$	
$\frac{1}{3}[T\times T]$	$\cong$	$\mathbb{Z}_2^4\rtimes\mathbb{Z}_3$	$\cong$	$\overline{[^+3,3,4^+]}$	
$\frac{1}{2}[\mathbb{D}_{2m}\times\mathbb{D}_{4n}]$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{D}_{2n}\big)^{\bullet}\mathbb{Z}_2$			$m,n\geq 2$
$\frac{1}{4}[\mathbb{D}_{4m}\times\mathbb{D}_{4n}]$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{Z}_n\big)\rtimes\mathbb{Z}_4$			m,n odd
$\frac{1}{2k}[\mathbb{D}_{2bk}\times\mathbb{D}_{2SK}]_s$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{Z}_n\big)\rtimes\mathbb{D}_{2k}$			$(s,k)=1$
$\frac{1}{2k}[\mathbb{D}_{2Ms}\times\mathbb{D}_{2nk}]_s$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{Z}_n\big)\rtimes\mathbb{D}_{2k}$			$(s,2k)=1; m,n$ odd
$\frac{1}{k}[\mathbb{Z}_{mk}\times\mathbb{Z}_{nk}]_s$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{Z}_n\big)_{\bullet}\mathbb{Z}_k$			$(s,k)=1$
$\frac{1}{k}[\mathbb{Z}_{mk}\times\mathbb{Z}_{nk}]_s$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{Z}_n\big)_{\bullet}\mathbb{Z}_k$			$(s,2k)=1; m,n$ odd
$\frac{1}{2}[\mathbb{D}_{2m}\times O]$	$\cong$	$\big(\mathbb{Z}_m\times T\big)\rtimes\mathbb{Z}_2$			
$\frac{1}{2}[\mathbb{D}_{4m}\times O]$	$\cong$	$\big(\mathbb{D}_{2m}\times T\big)\rtimes\mathbb{Z}_2$			$m\geq 2$
$\frac{1}{6}[\mathbb{D}_{6n}\times O]$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{Z}_2^2\big)\rtimes\mathbb{S}_3$			$m\geq 2$
$\frac{1}{2}[\mathbb{Z}_{2m}\times O]$	$\cong$	$\big(\mathbb{Z}_m\times T\big)_{\bullet}\mathbb{Z}_2$			split if $m=1$
$\frac{1}{3}[\mathbb{Z}_{3m}\times T]$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{Z}_2^2\big)_{\bullet}\mathbb{Z}_3$			split if $m=1$
$\frac{1}{2}[\mathbb{D}_{4m}\times\mathbb{D}_{4n}]$	$\cong$	$\big(\mathbb{D}_{2m}\times\mathbb{D}_{2n}\big)^{\bullet}\mathbb{Z}_2$			$m,n\geq 2$
$\frac{1}{2}[\mathbb{Z}_{2m}\times\mathbb{D}_{4n}]$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{D}_{2n}\big)^{\bullet}\mathbb{Z}_2$			$n\geq 2$
$\frac{1}{2}[\mathbb{Z}_{2m}\times\mathbb{D}_{2n}]$	$\cong$	$\big(\mathbb{Z}_m\times\mathbb{Z}_n\big)\rtimes\mathbb{Z}_2$	$\cong$	$\mathbb{Z}_m{\bullet}\mathbb{D}_{2n}$	

Table 6.5: Finite subgroups of $\mathrm{Aut}\big(\mathbb{P}^1\times\mathbb{P}^1\big)$ not conjugate to $A\times B$ for $A,B\leqslant\mathbb{PGL}_2(\mathbb{C})$.

Definition 202 ([DI10, Section 4.3])**.** We define the notation used in Table 6.5. The symbol $[p_1,\dots,p_r]$ refers to the Coxeter group defined by the Coxeter diagram and we write



$[p_1,\dots,p_r]^+$ to denote the index 2 subgroup of even length words in standard generators of the Coxeter group. If exactly one of the numbers $p_1,\dots,p_r$ is even, say p_k, there are two other subgroups of index 2 denoted by $[p_1,\dots,p_r^+]$ (resp. $[^+p_1,\dots,p_r]$). They consist of words which contain each generator $R_1,\dots,R_{k-1}$ (resp. $R_{k+1},\dots,R_r$) even number of times. The intersection of these two subgroups is denoted by $[^+p_1,\dots,p_r^+]$. For example,

$$\mathbb{D}_{2n}=[n],\ T=[3,3]^+,\ O=[3,4]^+,\ I=[3,5]^+.$$

We denote $\overline{[p_1,\dots,p_r]}$ the quotient of $[p_1,\dots,p_r]$ by its center.

We expect a similar result as Theorem 62 for $\mathbb{P}^1\times\mathbb{P}^1$, however at this time we have only a necessary condition. We leave for future work checking conditions (6.11) and the assumption of the non-existence of certain four point orbits of Theorem 203 against the above list of possible automorphism groups of $\mathbb{P}^1 \times \mathbb{P}^1$. We hope this scheme will allow us to answer Question B of Section 3.2 in full.

Theorem 203 (Cheltsov)**.** *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and G be a finite subgroup of* Aut(X)*. Suppose that for lines L_1 and L_2, one on each ruling, that*

$$\left.\begin{array}{r} |L_1|^G \\ |L_2|^G \\ |2L_1|^G \\ |2L_2|^G \\ |L_1+L_2|^G \\ |2L_1+L_2|^G \\ |L_1+2L_2|^G \\ |2L_1+2L_2|^G \end{array}\right\} = \emptyset. \tag{6.11}$$

In addition, assume that there is no orbit of exactly four points on X which imposes independent linear conditions on global sections of $H^0(X, L_1+L_2)$. Then

$$\operatorname{lct}(X,G) > 1.$$

Proof. In fact, if $\operatorname{lct}(X,G) > 1$, then $\operatorname{lct}(X,G) \geqslant \frac{3}{2}$. Let $\lambda < \frac{3}{2}$. Suppose there exists G-invariant divisor $D \equiv 2(L_1+L_2) \equiv -K_X$ such that the pair $(X, \lambda D)$ is not lc. Observe that for $n_i \geqslant 1$,

$$H^1\left(X, \mathcal{L}\left(X, \frac{3}{2}D\right) \otimes \left(-K_X + \frac{3}{2}D\right)\right) = H^1\left(X, \mathcal{L}\left(X, \frac{3}{2}D\right) \otimes (n_1L_1 + n_2L_2)\right) = 0,$$

where $\mathcal{L}\left(X, \frac{3}{2}D\right)$ is the sub-scheme of log canonical singularities of the pair $(X, \lambda D)$. By condition (6.11), $\mathcal{L}\left(X, \frac{3}{2}D\right)$ is O-dim.

We have the short exact sequence

$$0 \longrightarrow H^0\left(X, \mathcal{L}\left(X, \frac{3}{2}D\right) \otimes (L_1+L_2)\right) \longrightarrow H^0(X, L_1+L_2) \longrightarrow H^0\left(\mathcal{L}\left(X, \frac{3}{2}D\right), \mathcal{O}_{\mathcal{L}(X,\frac{3}{2}D)}\right) \longrightarrow 0.$$

Since $H^0\left(X, L_1+L_2\right) \cong \mathbb{C}^4$, Corollary 93 implies that there are four or less points in

$$\mathrm{LCS}(\mathrm{X}, \lambda \mathrm{D})=\operatorname{Supp}\left(\mathcal{L}\left(\mathrm{X}, \frac{3}{2} \mathrm{D}\right)\right).$$

In fact, there must be exactly four points in general position — however this contradicts our assumption that there is no orbit of exactly four points on X which imposes independent linear conditions on global sections of $H^0\left(X, L_1+L_2\right)$. Indeed, suppose that there is exactly

(i) one point. Then, as the representation of $G \leqslant \mathbb{PGL}_4(\mathbb{C})$ on $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ gives rise to a representation on $\mathbb{SL}_4(\mathbb{C}) \cong \mathbb{C}^4$, one point must correspond to an invariant line in $\mathbb{C}^4$. Thus, the representation on $\mathbb{C}^4$ splits as a direct sum of a one-dimensional and a three-dimensional one. However, the invariant three dimensional sub-representation corresponds to an invariant plane in $\mathbb{P}^3$ whose intersection with X is a member of $|L_1+L_2|^G$ — a contradiction.

(ii) two points. Similarly to the previous case, we must have an element in $|2L_1+2L_2|^G$ contradicting our assumptions.

(iii) three points. The three points lie on a plane in $\mathbb{P}^3$, which necessarily must be G-invariant — but this in then a member of $|L_1+L_2|^G$, a contradiction.

(iv) four points not in general position. If the points are not in general position then the lines that they lie on give rise to G-invariant elements in the empty linear systems (6.11). Specifically, there are four cases that, if excluded, we say that the four points are in generaln position:

(a) Four points in a 'square' arrangement produces an element in $|2L_1+2L_2|^G$.

(b) Three points on one line produces an element in $|L_1+L_2|^G$.

(c) Two points on a line from one ruling and two on a line from the other ruling yields an element in $|L_1+L_2|^G$.

(d) Two lines from the same ruling with two points each yields an element in $|2L_i|^G$, for $i=1,2$. □

6.9 The Projective Plane

A del Pezzo surface of degree nine is isomorphic to the projective plane $\mathbb{P}^2$. Its automorphism group is of course $\mathbb{PGL}_3(\mathbb{C})$. Assuming Conjecture 67 is true then groups $G \leqslant \mathbb{GL}_3(\mathbb{C})$, such that the quotient singularity $(V \ni P) = (\mathbb{C}^3 \ni 0)/G$ are exceptional correspond one-to-one with groups $\pi(G) \leqslant \mathbb{PGL}_3(\mathbb{C})$ such that $\operatorname{lct}(\mathbb{P}^2, \pi(G)) > 1$ (in fact $\operatorname{lct}(\mathbb{P}^2, \pi(G)) \geqslant \frac{4}{3}$ holds by Theorem 72), that is one-to-one to exceptional pairs $(\mathbb{P}^2, \pi(G))$. Here we write $\pi : \mathbb{GL}_3(\mathbb{C}) \longrightarrow \mathbb{PGL}_3(\mathbb{C})$ for the natural map from $\mathbb{GL}_3(\mathbb{C})$ to $\mathbb{PGL}_3(\mathbb{C})$. In this case, the exceptional version of Question B of Section 3.2 is answered completely by Markushevich-Prokhorov, ([MP99a, Theorems 1.2, 3.13 & Corollary 3.15]) and Cheltsov-Shramov ([CS09, Theorem 3.18]).

Proposition 204 ([MP99a, Corollary 3.15])**.** *Let G be a finite subgroup of* $\mathbb{GL}_3(\mathbb{C})$ *such that the quotient* $X = \mathbb{C}^3/G$ *is an exceptional singularity. Then, up to conjugation, G is one of the following subgroups of* $\mathbb{SL}_3(\mathbb{C})$*:*

- *Klein's simple group of order* 168, $\mathbb{PSL}_2(\mathbb{F}_7)$*;*
- *the unique central extension of Klein's simple group, (of order* 504*);*
- *the Hessian group (of order* 648*);*
- *the normal subgroup* F_{216} *of the Hessian group;*
- *or a central extension* I_{180} *of the alternating group* $\mathbb{A}_6$.

If Conjecture 67 turns out to be false then to complete the answer of Question B it is enough to identify those primitive groups G, who were omitted from the list of Markushevich and Prokhorov. Indeed, if G is imprimitive then it has a semi-invariant of degree three — that is a triple of lines in $\mathbb{C}^3$ permuted by G. Hence $\operatorname{lct}(\mathbb{P}^2, \pi(G)) \leqslant 1$ by Lemma 68. Primitive groups not included in Proposition 204 are those of types E and H in the notation of the Miller-Blichfeldt-Dickson classification of finite subgroups of $\mathbb{GL}_3(\mathbb{C})$ ([BDM16, Section 115]) — see the proof of Proposition 205 for a description of types E and H.

Using Lemma 68, the fact that groups of type H have a semi-invariant of degree three and those of type E have a semi-invariant of degree two we have the following.

Proposition 205. *Let G be a primitive subgroup of $\mathbb{GL}_3(\mathbb{C})$ and $\pi : \mathbb{GL}_3(\mathbb{C}) \longrightarrow \mathbb{PGL}_3(\mathbb{C})$ the natural map. Then*

$$\operatorname{lct}\big(\mathbb{P}^2, \pi(G)\big) \begin{cases} \leqslant & \frac{2}{3} \quad \textit{if } G \textit{ is of type } H \text{ — i.e. } \pi(G) \cong \mathbb{A}_5, \\ \leqslant & 1 \quad \textit{if } G \textit{ is of type } E \ (|\pi(G)| = 36), \\ > & 1 \quad \textit{otherwise.} \end{cases}$$

Proof. For full details on the groups of type E and H see [BDM16, Section 115]. We summarise below what is described in detail there.

For groups G of type H, $\pi(G)$ is the icosahedral group of order 60. Its three-dimensional representation has an invariant of degree two, which is a semi-invariant of degree two for any group of type H (cf. [MP99a, Theorem 3.13]).

For groups G of type E, $\pi(G)$ is a group of order 36. The group G is generated by the transformations

$$\operatorname{diag}(1, \epsilon_3, \epsilon_3^2), \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \rho & \rho & \rho \\ \rho & \rho\epsilon_3 & \rho\epsilon_3^2 \\ \rho & \rho\epsilon_3^2 & \rho\epsilon_3 \end{bmatrix},$$

where $\epsilon_3 = e^{\frac{2\pi i}{3}}$ and $\rho = \frac{1}{\epsilon_3 - \epsilon_3^2}$. Groups of type E, F and G leave a set of four triangles t_1, t_2, t_3, t_4 invariant in $\mathbb{C}^3$. $\pi(E)$ is completely characterised by its image in $\mathbb{S}_4$ where it is the subgroup of order two $\{1, (t_1 t_2)(t_3 t_4)\}$. For $i = 1, 2, 3, 4$, the t_i belong to one pencil of cubics, hence E acts on a pencil of plane cubics. That is, E acts on the pencil with image $\mathbb{Z}_2$ in $\operatorname{Aut}(\mathbb{P}^1)$ and since any involution on $\mathbb{P}^1$ has two fixed points, we have an invariant of degree two. Thus, any group of type E has a semi-invariant of degree two (cf. [MP99a, Lemma 3.14]).

The proof is now complete on applying Proposition 204 and Theorem 72. □

6.9.1 Further research

To answer completely our Question B, we need to identify those additional finite groups G such that $\operatorname{lct}\big(\mathbb{P}^2, \pi(G)\big) \geqslant 1$. A complete list of finite sub-groups of $\mathbb{PGL}_3(\mathbb{C})$ is well known — see for example [DI10, Section 4.2]. There the classification is split into three mutually exclusive groupings: intransitive; transitive and imprimitive; primitive (see Definition 63).

If G is intransitive, then this corresponds to a reducible representation. Hence we have a semi-invariant of degree one and by Lemma 68 $\left(\mathbb{P}^2, \pi(G)\right)$ is not weakly-exceptional

If G is transitive and imprimitive, then we have a semi-invariant of degree three and hence $\operatorname{lct}\left(\mathbb{P}^2, \pi(G)\right) \leqslant 1$. There are four types of group and we must check each to see if a global G-log canonical threshold of one is possible.

Theorem 206. *Let $\pi(G)$ be a transitive and imprimitive finite subgroup of $\mathbb{PGL}_3(\mathbb{C})$, then it is one of the following.*

- $\mathbb{Z}_n^3 \rtimes \mathbb{Z}_3$ *with generators*

$$[\epsilon_n x_0, x_1, x_2], [x_0, \epsilon_n x_1, x_2], [x_2, x_0, x_1];$$

- $\mathbb{Z}_n^3 \rtimes \mathbb{S}_3$ *with generators*

$$[\epsilon_n x_0, x_1, x_2], [x_0, \epsilon_n x_1, x_2], [x_0, x_2, x_1], [x_2, x_0, x_1];$$

- $\left(\mathbb{Z}_n \times \mathbb{Z}_{\frac{n}{k}}\right) \rtimes \mathbb{Z}_3$; *where $k > 1$, $k|n$ and $s^2 - s + 1 \equiv 1 \pmod k$; with generators*

$$[\epsilon_{\frac{n}{k}} x_0, x_1, x_2], [\epsilon_n^s x_0, \epsilon_n x_1, x_2], [x_0, x_2, x_1], [x_2, x_0, x_1];$$

- $\left(\mathbb{Z}_n \times \mathbb{Z}_{\frac{n}{3}}\right) \rtimes \mathbb{S}_3$ *with generators*

$$[\epsilon_{\frac{n}{3}} x_0, x_1, x_2], [\epsilon_n^2 x_0, \epsilon_n x_1, x_2], [x_0, x_2, x_1], [x_1, x_0, x_2];$$

In the last case where G is primitive, there are six types of group $E, F, \ldots, J$. The types F, G, I and J yield exceptional quotient singularities, and those of types E and H we checked above in Proposition 205.

As we mentioned in Chapter 3, recently Sakovics in [Sak10] proved the following theorem, extending the work of Markushevich-Prokhorov and answering completely our Question C in the weakly-exceptional case.

Theorem 207 ([Sak10, Theorem 1.20]). *Let $G \leqslant \mathbb{GL}_3(\mathbb{C})$ be a transitive finite subgroup without quasi-reflections. Then* $\operatorname{lct}\big(\mathbb{P}^2, \pi(G)\big) = 1$ *if, and only if, G is conjugate to a monomial group that is not isomorphic to a central extension of* $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_3$ *or* $(\mathbb{Z}_2)^2 \rtimes \mathbb{S}_3$.

Sketch. Using Theorem 74 it is enough to run through the Yau-Yu classification of finite subgroups of $\mathbb{SL}_3(\mathbb{C})$ and check those that admit semi-invariants of degree one or two. □

Chapter 7

Tables of Results

Below we include, for convenience, tables of the results of Chapter 6 on the calculations of the global G-invariant log canonical thresholds of smooth minimal del Pezzo G-surfaces $\left(S_d, G\right)$ where G is the full group of automorphisms of S_d. Firstly, we include Table 7.1 which answers Question A of Section 3.2 — that is, for a fixed S_d, when does there exist a finite group $G \leqslant \mathrm{Aut}(\mathrm{S_d})$ such that the pair $\left(S_d, G\right)$ is G-(weakly-)exceptional?

smooth del Pezzo surface of degree	1	2	3	4	5	6	7	$8_{\mathbb{F}_1}$	$8_{\mathbb{P}^1\times\mathbb{P}^1}$	9
weakly-exceptional	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	N	N	**Y**	**Y**
exceptional	**Y**	**Y**	**Y**	**Y**	**Y**	N	N	N	**Y**	**Y**

Table 7.1: Exceptional and weakly-exceptional smooth del Pezzo G-surfaces.

Smooth del Pezzo of degree five, S_5

$\mathrm{Aut}(\mathrm{S}_5)$	$\lvert\mathrm{Aut}(\mathrm{S}_5)\rvert$	Page ref.	$\mathrm{lct}\left(S_5, \mathrm{Aut}(\mathrm{S}_5)\right)$
$\mathbb{S}_5$	120	p 156	2
$\mathbb{A}_5$	60	p 155	2
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	20	p 154	1
$\mathbb{Z}_5 \rtimes \mathbb{Z}_2$	10	p 151	$\frac{4}{5}$
$\mathbb{Z}_5$	5	p 148	$\frac{4}{5}$
I	1	p 18	$\frac{1}{2}$

Table 7.2: Smooth del Pezzo surfaces of degree five.

Smooth del Pezzo of degree four, S_4

$\mathrm{Aut}(S_4)$	$\lvert\mathrm{Aut}(S_4)\rvert$	Page ref.	$\mathrm{lct}\big(S_4,\mathrm{Aut}(S_4)\big)$
$\mathbb{Z}_2^4 \rtimes \mathbb{D}_{10}$	160	p 145	2
$\mathbb{Z}_2^4 \rtimes \mathbb{S}_3$	96	p 143	2
$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$	64	p 143	1
$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	32	p 142	1
$\mathbb{Z}_2^4$	16	p 141	1
I	1	p 18	$\frac{2}{3}$

Table 7.3: Smooth del Pezzo surfaces of degree four.

Smooth del Pezzo of degree three, S_3

$\mathrm{Aut}(S_3)$	$\lvert\mathrm{Aut}(S_3)\rvert$	Page ref.	$\mathrm{lct}\big(S_3,\mathrm{Aut}(S_3)\big)$	Conditions
$\mathbb{Z}_3^3 \rtimes \mathbb{S}_4$	648	p 136	4	
$\mathbb{S}_5$	120	p 135	2	
$\mathbb{Z}_3\big(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4\big)$	108	p 134	1	
$\mathbb{Z}_3\big(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2\big)$	54	p 133	1	
$\mathbb{S}_4$	24	p 131	1	
$\mathbb{S}_3 \times \mathbb{Z}_2$	6	p 131	1	
$\mathbb{S}_3$	6	p 127	1	
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	p 127	$\frac{2}{3}$	
$\mathbb{Z}_8$	8	p 126	$\frac{2}{3}$	
$\mathbb{Z}_4$	4	p 125	$\frac{2}{3}$	
$\mathbb{Z}_2$	2	p 124	$\frac{2}{3}$	
I	1	p 18	$\frac{3}{4}$	S_3 has no Eckardt points
I	1	p 18	$\frac{2}{3}$	S_3 has an Eckardt point

Table 7.4: Smooth del Pezzo surfaces of degree three.

Smooth del Pezzo of degree two, S_2

$\mathrm{Aut}(S_2)$	$\lvert\mathrm{Aut}(S_2)\rvert$	Page ref.	$\mathrm{lct}\big(S_2, \mathrm{Aut}(S_2)\big)$	Conditions
$\mathbb{PSL}_2(\mathbb{F}_7)\times\mathbb{Z}_2$	336	p 119	2	
$\big(\mathbb{Z}_4^2\rtimes\mathbb{S}_3\big)\times\mathbb{Z}_2$	192	p 118	2	
$\mathbb{Z}_4{}_\bullet\mathbb{A}_4\times\mathbb{Z}_2$	72	p 117	1	
$\mathbb{S}_4\times\mathbb{Z}_2$	48	p 116	2	
$\big(\mathbb{D}_8\rtimes\mathbb{Z}_2\big)\times\mathbb{Z}_2$	32	p 115	1	
$\mathbb{Z}_{18}$	18	p 109	$\frac{3}{4}$	
$\mathbb{D}_8\times\mathbb{Z}_2$	16	p 114	1	
$\mathbb{S}_3\times\mathbb{Z}_2$	12	p 112	1	
$\mathbb{Z}_2\times\mathbb{Z}_6$	12	p 113	$\frac{3}{4}$	
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	6	p 111	1	
$\mathbb{Z}_6$	6	p 108	$\frac{3}{4}$	
$\mathbb{Z}_2\times\mathbb{Z}_2$	4	p 110	1	$\lvert -K_X\rvert^{\mathbb{Z}_2\times\mathbb{Z}_2}$ has no cusps or tacnodes
$\mathbb{Z}_2\times\mathbb{Z}_2$	4	p 110	$\frac{5}{6}$	$\lvert -K_X\rvert^{\mathbb{Z}_2\times\mathbb{Z}_2}$ contains cusps, no tacnodes
$\mathbb{Z}_2\times\mathbb{Z}_2$	4	p 110	$\frac{3}{4}$	$\lvert -K_X\rvert^{\mathbb{Z}_2\times\mathbb{Z}_2}$ contains tacnodes
$\mathbb{Z}_2$	2	p 107	1	$\lvert -K_X\rvert$ has no cusps or tacnodes
$\mathbb{Z}_2$	2	p 107	$\frac{5}{6}$	$\lvert -K_X\rvert$ contains cusps, no tacnodes
$\mathbb{Z}_2$	2	p 107	$\frac{3}{4}$	$\lvert -K_X\rvert$ contains tacnodes
I	1	p 18	$\frac{5}{6}$	$\lvert -K_X\rvert$ contains cusps, no tacnodes
I	1	p 18	$\frac{3}{4}$	$\lvert -K_X\rvert$ contains tacnodes

Table 7.5: Smooth del Pezzo surfaces of degree two.

Smooth del Pezzo of degree one, S_1

$\mathrm{Aut}(S_1)$	$\lvert\mathrm{Aut}(S_1)\rvert$	Page ref.	$\mathrm{lct}(S_1, \mathrm{Aut}(S_1))$	Conditions
$\mathbb{Z}_6 . \mathbb{D}_{12}$	72	p 97	2	
$\mathbb{Z}_3 \times \mathbb{Z}_2 . \mathbb{S}_4$	72	p 99	$\frac{5}{3}$	
$\mathbb{Z}_{30}$	30	p 77	$\frac{5}{6}$	
$\mathbb{Z}_2 . \mathbb{D}_{12}$	24	p 90	2	
$\mathbb{Z}_3 \times \mathbb{D}_8$	24	p 94	$\frac{5}{3}$	
$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	24	p 75	1	
$\mathbb{Z}_{24}$	24	p 76	$\frac{5}{6}$	
$\mathbb{Z}_2 . \mathbb{A}_4$	20	p 92	2	
$\mathbb{Z}_{20}$	20	p 74	$\frac{5}{6}$	
$\mathbb{Z}_2 \times \mathbb{Z}_3 . \mathbb{D}_6$	18	p 95	2	
$\mathbb{D}_{16}$	16	p 87	$\frac{5}{3}$	
$\mathbb{D}_{12}$	12	p 84	2	
$\mathbb{Z}_{12}$	12	p 73	$\frac{5}{6}$	
$\mathbb{Z}_2 \times \mathbb{Z}_6$	12	p 71	1	
$\mathbb{Z}_{10}$	10	p 70	$\frac{5}{6}$	
$\mathbb{D}_8$	8	p 77	2	
$\mathbb{D}_8$	8	p 77	$\frac{5}{3}$	$X = \{t^2 + z^3 + bzx^2y^2 + xy(c(x^4 + y^4) + dx^2y^2) = 0\}$ with $c \neq 0$ and $d \neq 0$
$\mathbb{Z}_2 . \mathbb{D}_4$	8	p 82	2	
$\mathbb{Z}_2 . \mathbb{D}_4$	8	p 82	$\frac{5}{6}$	$X = \{t^2 + z^3 + z(a(x^4 + y^4) + bx^2y^2) + xy(x^4 - y^4) = 0\}$ with $a = 0$ or $2(1+\epsilon_4^k) + \epsilon_4^{2k} b = 0$
$\mathbb{Z}_2 . \mathbb{D}_4$	8	p 82	$\frac{5}{3}$	$X = \{t^2 + z^3 + z(a(x^4 + y^4) + bx^2y^2) + xy(x^4 - y^4) = 0\}$ with $2a \pm b = 0$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	8	p 68	1	
$\mathbb{Z}_8$	8	p 69	$\frac{5}{6}$	
$\mathbb{Z}_6$	6	p 65	1	
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	p 63	1	
$\mathbb{Z}_4$	4	p 64	1	
$\mathbb{Z}_4$	4	p 64	$\frac{5}{6}$	$X = \{t^2 + z^3 + z(ax^4 + bx^2y^2 + cy^4) + xy(dx^4 + ex^2y^2 + fy^4) = 0\}$ with $a = 0$ or $c = 0$ (but not $a = 0$, $c = 0$ and $d = f$ concurrently)
$\mathbb{Z}_2$	2	p 63	1	$\lvert -K_X \rvert$ has no cusps
$\mathbb{Z}_2$	2	p 63	$\frac{5}{6}$	$\lvert -K_X \rvert$ contains cusps
I	1	p 18	1	$\lvert -K_X \rvert$ has no cusps
I	1	p 18	$\frac{5}{6}$	$\lvert -K_X \rvert$ contains cusps

Table 7.6: Smooth del Pezzo surfaces of degree one.

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