1S Calculus

Sections 1.14 – 1.15

University of Glasgow

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1.14 Logarithmic integrals

Theorem

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c.$$

Example

Find the following indefinite and definite integrals

i) $\displaystyle\int \frac{\cos x}{5+\sin x}\,dx,$

ii) $\displaystyle\int \frac{3x}{x^2+1}\,dx,$

iii) $\displaystyle\int \frac{4x-3}{x^2-4x+13}\,dx,$

iv) $\int_0^{\sqrt{2}} \frac{3x+4}{x^2+2}\,dx.$

Example

Verify the following Standard Integrals.

$$\int \cot x \, dx = \log|\sin x| + c,$$

and

$$\int \tan x \, dx = \log|\sec x| + c.$$

1.15 Integration of rational functions

Definition (Rational function)

A rational function $r(x)$ is the ratio of two polynomial functions

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. If $\deg p < \deg q$ then $r(x)$ is a *proper* rational function, otherwise it is *improper*. Improper rational functions can be written as the sum of a polynomial and a proper rational function, via polynomial long division.

Example

A proper rational function

$$\frac{x^2 + 3x + 1}{x^3 - 5x^2 + 3}.$$

Example (Rational functions and integration)

The following examples use rational functions.

i) Write the improper rational function

$$\frac{x^4 + x^3 - 2x^2 + 21x - 6}{x^2 + 3x - 1}$$

as the sum of a polynomial and a proper rational function.

ii) What is $\displaystyle\int \frac{x^2}{x+1}\,dx$?

Technique (Partial Fractions)

Given a proper rational function $\frac{p(x)}{q(x)}$, we can factor $q(x)$ into a product of irreducible polynomials. These polynomial factors are either linear or irreducible quadratics. The table below relates the irreducible factors to the partial fractions.

Factorisation produces:	Partial fractions
Distinct linear factors $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ $\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_n$	are $\frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_n}{x-\alpha_n}$ $A_1, A_2, \ldots, A_n$ constants
Repeated factors e.g. $(x-\alpha)^r$, $r \geqslant 2$	include r terms e.g. $\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_r}{(x-\alpha)^r}$
Irreducible quadratic factor (has no real roots) e.g. x^2+px+q	include $\frac{Ax+B}{x^2+px+q}$

Example (Use of partial fractions in integration)

i) $\int \frac{9x+3}{(x-1)(x+2)(x-3)}\,dx$

ii) $\int \frac{1}{x^2-a^2}$, where $a > 0$ is a constant.

iii) $\int \frac{x^2-4x+15}{(x-1)^2(x+2)}$

iv) $\int \frac{x^2+2x-17}{(x-3)(x^2-6x+10)}$