1S Calculus

Chapter 5 – Limits

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You should be familiar with the following results

- For any $\alpha > 0$, $\dfrac{1}{x^\alpha} \to 0$ as $x \to \infty$.
- For any fixed $x \in (0,1)$, $x^n \to 0$ as $n \to \infty$.
- $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1.$
- $\tan x = \dfrac{\sin x}{\cos x}$ and at $x = \dfrac{\pi}{2}$ there is a vertical asymptote.

$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}.$$

Similarly,

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

5.1 Revision

- Tug of war between $\exp$ and $\lim$:

Function	Potency in limits
Exponential	Dominant (Always wins)
Powers of x (polynomials)	Middling
Logarithms	Feeble (Always loses)

- $$0 < \log x < x \text{ for all } x > 1$$
- For $\alpha > 0$,
$$\frac{\log x}{x^\alpha} \to 0 \text{ as } x \to \infty$$
- For any constant α,
$$x^\alpha e^{-x} \to 0 \text{ as } x \to \infty$$

As $x \to \infty$ the **exponential dominates** the polynomial x^2.

5.2 Improper integrals of the first kind

Consider integrals of the form $\int_a^{\infty} f(x)\,dx$ (or $\int_{-\infty}^{a} f(x)\,dx$).

Definition

The improper integral is defined as

$$\int_a^{\infty} f(x)\,dx = \lim_{M\to\infty} \int_a^{M} f(x)\,dx$$

if the limit exists. Similarly, $\int_{-\infty}^{a} f(x)\,dx = \lim_{M\to-\infty} \int_M^{a} f(x)\,dx.$

Example (Evaluate the following improper integrals)

i) $\int_2^{\infty} \frac{1}{x^2}\,dx.$

ii) $\int_2^{\infty} \frac{1}{\sqrt{x}}\,dx.$

5.2 Improper integrals of the first kind

Example (Evaluate the following improper integrals)

i) $\displaystyle\int_1^\infty \frac{1}{x^5}\,dx$

ii) $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx$

iii) $\int_0^\infty xe^{-2x}\,dx$

iv) $\displaystyle\int_0^\infty \frac{dx}{(x^2+4)^2}$

Example (Gabriel's horn)

Calculate the volume of revolution V_M for the curve $y = x^{-1}$, revolved around the x-axis between $x = 1$ and $x = M$, calculate the curved surface area A_M for the same volume of revolution. Take the limit $M \to \infty$, do V_M and A_M have limits? Could you finish a drink from Gabriel's horn? Could you paint the inside of the horn? Is this a paradox?