Smooth Exceptional del Pezzo Surfaces

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For a Fano variety V with at most Kawamata log terminal (klt) singularities and a finite group G acting bi-regularly on V, we say that V is *G-exceptional* (resp., *G-weakly-exceptional)* if the log pair (V,Δ) is klt (resp., log canonical) for all G-invariant effective $\mathbb{Q}$-divisors Δ numerically equivalent to the anti-canonical divisor of V. Such G-exceptional klt Fano varieties V are conjectured to lie in finitely many families by Shokurov ([Sho00, Pro01]). The only cases for which the conjecture is known to hold true are when the dimension of V is one, two, or V is isomorphic to n-dimensional projective space for some n. For the latter, it can be shown that G must be primitive — which implies, in particular, that there exist only finitely many such G (up to conjugation) by a theorem of Jordan ([Pro00]).

Smooth G-weakly-exceptional Fano varieties play an important role in non-rationality problems in birational geometry. From the work of Demailly (see [CS08, Appendix A]) it follows that Tian's α_G-invariant for such varieties is no smaller than one, and by a theorem of Tian such varieties admit G-invariant Kähler-Einstein metrics. Moreover, for a smooth G-exceptional Fano variety and given any G-invariant Kähler form in the first Chern class, the Kähler-Ricci iteration converges exponentially fast to the Kähler form associated to a Kähler-Einstein metric in the $C^\infty(V)$-topology. The term *exceptional* is inherited from singularity theory, to which this study enjoys strong links.

We classify two-dimensional smooth G-exceptional Fano varieties (del Pezzo surfaces) and provide a partial list of all G-exceptional and G-weakly-exceptional pairs (S,G), where S is a smooth del Pezzo surface and G is a finite group of automorphisms of S. Our classification confirms many conjectures on two-dimensional smooth exceptional Fano varieties.

References

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