# 1S Calculus

Week 17

University of Glasow

Monday, $7^{\text{th}}$ January 2013

Keith Devlin – Calculus: One of the Most Successful Technologies


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|---|---|---|
| ◦ | Integration | 60% |
| ◦ | Ordinary Differential Eqs. | 15% |
| ◦ | Maclaurin Series | 15% |
| ◦ | Numerical Methods | 5% |
| ◦ | Limits | 5% |

In this course we introduce integration as the *inverse* of differentiation.

**Question**

*Given the derivative of some unknown function, what can you say about that function?*

That is,

**Question**

*Given that* $\dfrac{dy}{dx} = f(x)$*, then what can be said about* $y(x)$*?*

# 1.1 Antidifferentiation

## Definition

*For a given $f(x)$, $F(x)$ is an antiderivative of $f(x)$ on the interval $I$ if*

$$\frac{d}{dx}F(x) = f(x) \quad \text{for } x \in I.$$

## Example

The function $\frac{1}{3}x^3$ is an antiderivative of $x^2$ (on $\mathbb{R}$).

## Theorem

i) *If $F(x)$ is an antiderivative of $f(x)$ on the interval $I$, then $F(x) + c$ is also an antiderivative for any constant $c$.*

ii) *Any antiderivative of $f(x)$ can be written $F(x) + d$ for some constant $d$.*

**Example**

Antiderivatives of $\cos x$ and $x^{-1}$.

$$\int \cos x \, dx = \sin x + c \qquad \text{(SI 3)}$$

$$\int \frac{1}{x} \, dx = \log |x| + c \qquad \text{(SI 8)}$$

Refer to the list of standard integrals (SI) in Chapter 6 (p 69).

**Example**

Verify SI 9, $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c, \quad a > 0.$

**Exercise**

*Verify some of the remaining standard integrals, especially SI 1.*

## Theorem

*Integration is linear (the statements below should be interpreted as addition of equivalence classes)*

i) $\displaystyle\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx.$

ii) *For a constant* $k$, $\displaystyle\int k f(x)\, dx = k \int f(x)\, dx.$

## Example

Using SI 1, SI 4 and SI 10 (p 69 in Chapter 6)

i) Integration of a constant, $\int k\, dx$.

ii) Integration of powers of $x$,

   a) $\int (x^4 + 3\sqrt{x} - 5)\, dx$.

   b) $\int \left(3\sec^2 x - \frac{4}{x^2+4}\right) dx$.

**Example (using antidifferentiation to solve an *ordinary differential eqn*)**

Given $x \in (0, \infty)$ and

$$\frac{d^2y}{dx^2} = \frac{4}{x^3} - \frac{1}{x^2},$$

where $y'(1) = 0$ and $y(1) = 0$ find $y(x)$.

**Example**

i) Integration of powers of $x$,
   a) $\int \sqrt{x}(x^2 + 2)\, dx$,
   b) $\int \frac{(1-\sqrt{x})^2}{x}\, dx$,
   c) $\int \left(x + \frac{1}{x}\right)^2\, dx$,

ii) What is $\displaystyle\int \frac{1}{1 + \sin x}\, dx$?

# 1.3 The $px+q$ rule

**Theorem (The $px+q$ rule)**

*Let $p, q$ be constants with $p \neq 0$. Suppose $F(x)$ is the antiderivative of $f(x)$, then*

$$\int f(px+q)\,dx = \frac{1}{p}F(px+q)+c.$$

**Example (Applications of the '$px+q$' rule)**

i) $\displaystyle\int \cos(3x+4)\,dx$

ii) $\displaystyle\int \sec^2(7x-3)\,dx$

iii) $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx$

iv) $\displaystyle\int \frac{1}{9x^2+16}\,dx$

v) $\displaystyle\int \cos 2x\,dx$

vi) $\displaystyle\int \sin 3x\,dx$

vii) $\displaystyle\int e^{4x}\,dx$

# 1.3 The $px + q$ rule

In general we have,

**Lemma**

$$\int \cos px \, dx = \frac{1}{p} \sin px + c, \qquad \int \sin px \, dx = -\frac{1}{p} \cos px + c,$$

$$\int e^{px} \, dx = \frac{1}{p} e^{px} + c.$$

**Example (Further examples of the '$px + q$' rule)**

i) $\int (3x+7)^4 \, dx$

ii) $\int \frac{1}{(5-2x)^2} \, dx$

iii) $\int (x+4)^7 \, dx$

iv) $\int \frac{1}{5-3x} \, dx$

v) $\int \frac{1}{x+2} \, dx$

vi) $\int 2^x \, dx$

# 1.4 Integration of squares of trigonometric functions

**Lemma**

i) $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c,$

ii) $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c,$

iii) $\int \tan^2 x\, dx = \tan x - x + c.$

**Example (Using trigonometric identities and the '$px + q$' rule)**

i) $$\int \tan^2(3x)\, dx$$

ii) $$\int \sin^2(2x)\, dx$$

iii) $$\int \frac{1}{1 + \cos x}\, dx$$

## Example

Integration of $\dfrac{1}{\text{irreducible quadratic}}$.

i) $\int \frac{1}{x^2+6x+13}\,dx$

ii) $\int \frac{1}{x^2+x+1}\,dx$

## Example

Integration of $\dfrac{1}{\sqrt{\text{quadratic}}}$ with quadratic term $-x^2$.

i) $\int \frac{1}{\sqrt{8-2x-x^2}}\,dx$

ii) $\int \frac{1}{\sqrt{1+4x-x^2}}\,dx$