1S Calculus

Sections 1.8 – 1.11

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1.8 Definite integrals

Notation: For any function F, $[F(x)]_a^b = F(b) - F(a)$.

Definition

The expression

$$\int_a^b f(x)\,dx$$

is the **definite integral** of $f(x)$ from a to b. It is defined as $[F(x)]_a^b$ where $F(x)$ is an antiderivative of $f(x)$ and a and b are called the **limits** of integration.

Example (Evaluate the following definite integrals)

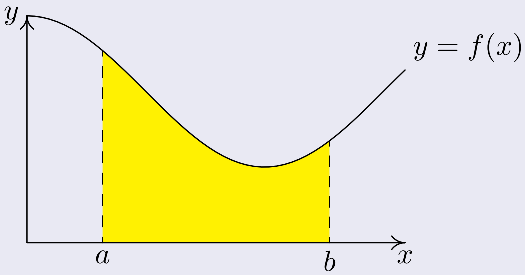
i) $\displaystyle\int_3^6 \frac{1}{x}\,dx$

ii) $\int_0^1 \frac{dx}{x^2+3}$

iii) $\int_0^{\pi/4} \sin 2x\,dx$

Theorem (The Fundamental Theorem of Integral Calculus)

For a function f which is continuous on the interval $[a, b]$, the area of the region R indicated in the sketch below (the area between the graph of f, the x-axis and the lines $x = a$, $x = b$) is $\int_a^b f(x)\,dx$.



Area under the curve $y = f(x)$ between $x = a$ and $x = b$.

Theorem (Properties of definite integrals)

Definite integrals have the following properties.

i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ *and* $\int_a^a f(x)\,dx = 0.$

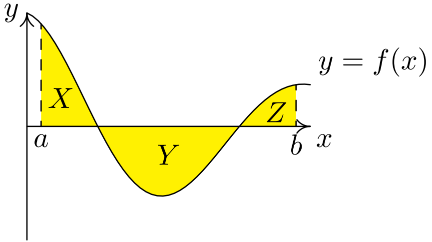
ii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad (a \leqslant c \leqslant b).$

iii) **Integral of a constant:** *For $k > 0$ a constant* $\int_a^b k\,dx = k\,(b-a)$*, the area of a rectangle whose sides are length k and $b-a$.*

iv) **Area between two curves:** *The area between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is*

$$\int_a^b f(x) - g(x)\,dx.$$

Functions that change sign e.g. for the situation illustrated,



$$\int_a^b f(x)\,dx = \text{area X} - \text{area Y} + \text{area Z}.$$

Example

i) Find the area between the x-axis and the curve $y = \sin^2 x$, and the lines $x = 0$ and $x = \pi/2$.

ii) Find the area of the finite region bounded by the curves $y = \sqrt{x}$ and $y = \dfrac{1}{8}x^2$.

1.11 Change of variables in a definite integral

Recall that if $F(x)$ is an antiderivative of $f(x)$, then $F\big(g(x)\big)$ is an antiderivative of $f\big(g(x)\big) \cdot g'(x)$.

$$\int_a^b f(g(x))g'(x)\,dx = [F(g(x))]_{x=a}^{x=b} = F(g(b)) - F(g(a))$$

$$= [F(u)]_{u=g(a)}^{u=g(b)}$$

$$= \int_{g(a)}^{g(b)} f(u)\,du.$$

When we change variable we must remember to also change the limits from $x = a$ to $x = b$ to $u = g(a)$ to $u = g(b)$.

1.11 Change of variables in a definite integral

Example (Calculate the following definite integrals using substitution)

i) $\displaystyle\int_0^{\pi/3} \tan^3 x \sec^2 x \, dx.$

ii) $\displaystyle\int_0^{4} x\sqrt{x^2+9} \, dx.$

iii) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sqrt{3+\sin^2 x}} \, dx.$