1S Calculus

Sections 1.12 – 1.15

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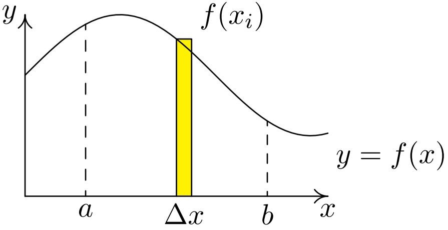
January 2013

An approximation to the area under the graph of $y = f(x)$ is the sum of the areas of the rectangles.



$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{i=n-1} f(x_i)\Delta x.$$

The area under f between a and b is the limit of this sum as $n \to \infty$,

i.e. $$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0}\left[\sum_{i=0}^{i=n-1} f(x_i)\Delta x\right] = \lim_{n\to\infty}\left[\sum_{i=0}^{i=n-1} f(x_i)\Delta x\right].$$

- Let V be the volume of a solid object generated by taking the area under the curve $y = f(x)$ above the interval $[a, b]$ and rotating it about the x-axis through a complete revolution.
- The volume of a small slice of V at location x_i can be approximated by the volume of a cylinder of radius $f(x_i)$ and length Δx.
- The volume of this cylindrical region is $\pi f(x_i)^2 \Delta x$. So

$$V \approx \sum_{i=0}^{n-1} \pi [f(x_i)]^2 \Delta x.$$

- The volume of V is the limit of the sum

$$V = \lim_{n \to \infty} \sum_{i=0}^{n-1} \pi [f(x_i)]^2 \Delta x = \pi \int_a^b [f(x)]^2 \, dx.$$

Example

i) Show that the volume of a circular cone of height h and base radius r is $\frac{1}{3}\pi r^2 h = \frac{1}{3} \times$ base $\times$ height.

ii) Calculate the volume of the finite region of the solid of revolution generated by rotating the curve $y = 1 - x^4$ about the x-axis.

1.12.2 Area of a surface of revolution

Let A be the area of the curved surface generated by revolving the graph of $y = f(x)$ around the x-axis, between $x = a$ and $x = b$.

$$A \approx \sum_{i=0}^{i=n-1} 2\pi f(x_i)\sqrt{1 + f'(x_i)^2}\Delta x$$

The area is the limit of the sum, hence

$$A = \lim_{n\to\infty} \sum_{i=0}^{n-1} 2\pi f(x_i)\sqrt{1 + f'(x_i)^2}\Delta x = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx.$$

Example

Find the area of the curved surface formed by rotating $y = \sqrt{x}$ about the x-axis, between $x = 0$ and $x = 2$.

1.13 Integration by parts

Theorem (Integration by parts)

$$\int f'(x) \cdot g(x)\, dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)\, dx$$

For **definite integrals** integration by parts is written

$$\boxed{\int_a^b f'(x) \cdot g(x)\, dx = [f(x) \cdot g(x)]_a^b - \int_a^b f(x) \cdot g'(x)\, dx.}$$

Example (Find the following indefinite integrals)

i) $\displaystyle\int x \cdot \cos 2x\, dx$

ii) $\displaystyle\int x \cdot e^{3x}\, dx$

iii) $\displaystyle\int x^3 \cdot \log x\, dx$

Example (Find the following integrals)

iv) $\int x^2 \cdot \sin x \, dx$

v) $\int_0^1 x \cdot e^{2x} \, dx$

vi) $\int_0^\pi x^2 \cdot \cos 2x \, dx$

Technique (A)

Take $f(x) = 1$ *and apply integration by parts.*

Example (Apply Technique A)

i) $\int \log x \, dx$

ii) $$\int \sin^{-1} x \, dx$$

Technique (B)

Get the original integral to reappear on the RHS after integrating by parts.

Example

Show that

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log |\sec x + \tan x| + c$$

by applying Technique B.

Example (Apply Technique B to solve the following integrals)

i) $\displaystyle\int e^x \sin x \, dx$,

ii) $\displaystyle\int e^x \cos x \, dx$.