1S Calculus

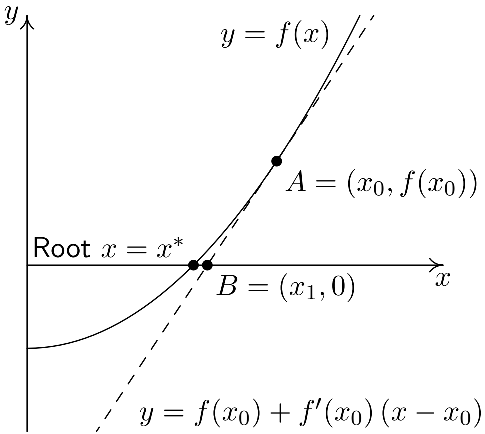
Chapter 3 – Numerical Methods

University of Glasgow

March 2013

4.1 Newton–Raphson: Nonlinear root finding

Suppose we are trying to solve $f(x) = 0$, i.e. finding the roots of f.



Technique

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}; \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x^3 - x - 5 = 0$.

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Find a root of $f(x) = x^3 - x - 5 = 0$.

- $f(1) = 1^3 - 1 - 5$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x^3 - x - 5 = 0$.

- $f(1) = -5$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x^3 - x - 5 = 0$.

- $f(1) < 0$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x^3 - x - 5 = 0$.

- $f(1) < 0$ and $f(2) = 2^3 - 2 - 5$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x^3 - x - 5 = 0$.

- $f(1) < 0$ and $f(2) = 1 > 0$

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- $f(1) < 0$ and $f(2) > 0$, hence the root lies in the interval $(1, 2)$.

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Find a root of $f(x) = x^3 - x - 5 = 0$.

- $f(1) < 0$ and $f(2) > 0$, hence the root lies in the interval $(1, 2)$.
- Start with $x_0 = 2$ as $f(2)$ is closer to zero than $f(1)$.

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  we have $$x_{n+1} = x_n - \frac{x_n^3 - x_n - 5}{3x_n^2 - 1}$$

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  $$x_1 = \frac{2x_0^3 + 5}{3x_0^2 - 1}$$

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- The sequence $x_0, x_1, x_2, \ldots$ is converging and so 1.90416086 is an approximation to root of $f(x) = 0$ in $[1, 2]$ to 8 dp.

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- The sequence $x_0, x_1, x_2, \ldots$ is converging and so 1.90416086 is an approximation to root of $f(x) = 0$ in $[1, 2]$ to 8 dp.

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x - e^{-x} = 0$.

- $f(0) = 0 - e^0$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x - e^{-x} = 0$.

- $f(0) = -1$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x - e^{-x} = 0$.

- $f(0) < 0$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x - e^{-x} = 0$.

- $f(0) < 0$ and $f(1) = 1 - e^{-1}$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x - e^{-x} = 0$.

- $f(0) < 0$ and $f(1) \simeq 1 - 0.367879$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x - e^{-x} = 0$.

- $f(0) < 0$ and $f(1) > 0$

Example (Use the N-R method to find an approx. to the root)

Find a root of $f(x) = x - e^{-x} = 0$.

- $f(0) < 0$ and $f(1) > 0$, hence the root lies in the interval $(0, 1)$.

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Find a root of $f(x) = x - e^{-x} = 0$.

- $f(0) < 0$ and $f(1) > 0$, hence the root lies in the interval $(0, 1)$.
- Start with $x_0 = 1$.

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- Using the N-R method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $\quad f'(x) = 1 + e^{-x}$, we have

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

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$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \frac{e^{-x_n}(x_n + 1)}{1 + e^{-x_n}}$$

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$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \frac{e^{-x_n}(x_n + 1)}{1 + e^{-x_n}} = \frac{1 + x_n}{1 + e^{x_n}}.$$

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- Repeated application gives the sequence

$$x_1 = \frac{1 + x_0}{1 + e^{x_0}}$$

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- The sequence is converging and so 0.56714329 is an approximate to root of $f(x) = 0$ in $[0, 1]$ to 8 dp.

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4.2 Simpson's rule

- We know $I = \int_a^b f(x)\,dx$ corresponds to the area under the curve $y = f(x)$ from a to b,
- hence $I = F(b) - F(a)$, where F is an antiderivative of f.
- However, in practice, computing an antiderivative for f may be (fiendishly) difficult (or f may have been derived from an approximation).
- An alternative is to compute I numerically.

Idea

Approximate f by functions that are easy to integrate.

We can approximate f

- with rectangles,
- with trapezoids,
- with polynomials.

Theorem

If f is a polynomial of degree $\leqslant 3$ (i.e. $f(x) = px^3 + qx^2 + rx + s$, where p, q, r, s are constants), then

$$\int_\alpha^\beta f(x)\,dx = \frac{\beta - \alpha}{6}(f(\alpha) + 4f(\mu) + f(\beta)),$$

where $\mu = (\alpha + \beta)/2$ is the midpoint of the interval $[\alpha, \beta]$.

Hence, **Simpson's rule** is given by:

$$\int_a^b f(x)\,dx = \frac{b-a}{6n}\Bigg(f(c_0) + f(c_{2n}) + 4\underbrace{(f(c_1) + f(c_3) + \cdots + f(c_{2n-1}))}_{\text{sum of interior \textbf{odd} locations}} + 2\underbrace{(f(c_2) + f(c_4) + \cdots + f(c_{2n-2}))}_{\text{sum of interior \textbf{even} locations}}\Bigg).$$

Example (Simpson's rule)

i) Use Simpson's Rule to approximate $\int_1^2 \frac{1}{x}\, dx$ by dividing $[1, 2]$ into 4 equal subintervals.

$$\int_1^2 \frac{1}{x}\, dx = 0.6933 \quad \text{to 4dp.}$$

The true value of $\int_1^2 \frac{1}{x}\, dx$ is $\log 2 \approx 0.693147$.

ii) $\int_0^1 e^{x^2}\, dx$, dividing $[0, 1]$ into 4 equal parts.

$$\int_0^1 e^{x^2}\, dx = 1.46371076044 \text{ (11 dp)}$$

The true value of $\int_0^1 e^{x^2}\, dx$ is 1.46265174590 (11 dp).